COMS20010 — Problem sheet 1

This problem sheet covers week 1, focusing on induction and O-notation. You don't have to finish the problem sheets before the class — focus on understanding the material the problem sheet is based on. If working on the problem sheet is the best way of doing that, for you, then that's what you should do, but don't be afraid to spend the time going back over the quiz and videos instead. (Then come back to the sheet when you need to revise for the exam!)

As with the Blackboard quizzes, question difficulty is denoted as follows:

- $\star$ You'll need to understand facts from the lecture notes.
- $\star\star$ You'll need to understand and apply facts and methods from the lecture notes in unfamiliar situations.
- $\star\star\star$ You'll need to understand and apply facts and methods from the lecture notes and also move a bit beyond them, e.g. by seeing how to modify an algorithm.
- $\star\star\star\star$ You'll need to understand and apply facts and methods from the lecture notes in a way that requires significant creativity. You should expect to spend at least 5–10 minutes thinking about the question before you see how to answer it, and maybe far longer. Only 20% of marks in the exam will be from questions set at this level.
- $\star\star\star\star\star$ These questions are harder than anything that will appear on the exam, and are intended for the strongest students to test themselves. It's worth trying them if you're interested in doing an algorithms-based project next year — whether you manage them or not, if you enjoy thinking about them then it would be a good fit.

1 Big-O Notation

1. (a) [$\star$] Show that $5n \in O(n^2)$ from the definition.

   Solution: Pick $C = 1$, $n_0 = 5$. Then, for all $n \geq n_0 = 5$ we have $5n \leq n^2 = C \cdot n^2$.

   (b) [$\star\star$] Show that $2n^2 + 3n + 1 \in O(n^2 + 5n + 4)$ from the definition.

   Solution: Pick $C = 5$, $n_0 = 1$. Then, we have

   $$2n^2 + 3n + 1 \leq 5 \cdot (n^2 + 5n + 4) = 5n^2 + 25n + 20.$$

   And this clearly holds for all $n \geq n_0 = 1$.

   (c) [$\star\star\star$]

   i. Show that

   $$\sum_{i=1}^{n} i^4 \in O(n^5).$$

   Solution: We will answer this by finding an upper bound for the left-hand side, noting that it doesn't matter how loose our bound is.

For each i^4 term in the sum we have $i \le n$, and so $i^4 \le n^4$. This gives us

$$\sum_{i=1}^{n} i^4 \le \sum_{i=1}^{n} n^4 = n \cdot n^4 = n^5.$$

We now use the standard property of O-notation that if $f(n) \le g(n)$ for all sufficiently large n, then $f(n) \in O(g(n))$; this follows from the definition of O-notation, and is mentioned on the week's quiz.

ii. Show that

$$\sum_{i=1}^{n} i^4 \in \Omega(n^5).$$

Solution: We discard the smaller terms in the sum, then bound the rest below to get

$$\sum_{i=1}^{n} i^4 \ge \sum_{i=\lceil n/2 \rceil}^{n} i^4 \ge \sum_{i=\lceil n/2 \rceil}^{n} \lceil n/2 \rceil^4 \ge (n - \lceil n/2 \rceil + 1)\lceil n/2 \rceil^4.$$

We need to be a little careful here because of the ceilings — it's easy to accidentally write something that's false for $n = 1$, say. However, it is always true that $x + 1 \ge \lceil x \rceil \ge x$ for all $x \ge 0$. Thus

$$\sum_{i=1}^{n} i^4 \ge \big(n - \lceil n/2 \rceil + 1\big)\lceil n/2 \rceil^4 \ge (n/2)^5 \ge n^5/32.$$

Since $n^5/32 \in \Omega(n^5)$, the result again follows from standard properties of O-notation. Notice that there's nothing magical about splitting the sum at $i = \lceil n/2 \rceil$; we could split it at $i = \lceil \alpha n \rceil$ or $\lfloor \alpha n \rfloor$ for any $\alpha \in (0, 1)$ and a very similar argument would still have worked, with slightly different constants. The key point is: when you're trying to bound a sum below, and you don't care about constant factors, feel free to throw half the terms away.

iii. Show that

$$\sum_{i=1}^{n} i^4 \in \Theta(n^5).$$

Hint: What does Θ mean in terms of O and Ω?

Solution: We know $f(n) \in \Theta(g(n))$ if and only if $f \in O(g(n))$ and $f \in \Omega(g(n))$. The result then immediately follows from parts (i) and (ii).

(d) [$\star\star$] Show that $3^n \in 2^{O(n)}$.

Solution: We have

$$3^n = 2^{n \log_2(3)} \in 2^{O(n)}.$$

(e) [$\star\star$] Show that $n + \log n \in O(n)$.

Solution: Here we will use the the general property that if $f(n) \in O(h(n))$ and $g(n) \in O(h(n))$ then

$$O(f(n) + g(n)) \in O(h(n)).$$

(This is shown in the Blackboard quiz.) Observe that $n \in O(n)$ and $\log(n) \in O(n)$, which gives us

$$O(n + \log(n)) \in O(n).$$

2. Why don't the following alternative definitions for O-notation (denoted by $O^\dagger$) mean the same thing as the usual definition? For each one, give an example of functions f and g such that $f(n) \in O(g(n))$ but $f(n) \notin O^\dagger(g(n))$, or such that $f(n) \in O^\dagger(g(n))$ but $f(n) \notin O(g(n))$.
   (a) [$\star\star$] $f(n) \in O^\dagger(g(n))$ if there exists $n_0 > 0$ such that for all $n \geq n_0$, $f(n) \leq g(n)$.

   Solution: Take g to be smaller than f by a constant factor. For example, $n \in O(n/3)$, but $n \notin O^\dagger(n/3)$, since $n > n/3$ for all $n \geq 1$.

   (b) [$\star\star$] $f(n) \in O^\dagger(g(n))$ if there exists $C > 0$ such that $f(n) \leq Cg(n)$ for all $n \geq 0$.

   Solution: Take g to grow faster than f, but be negative or zero for some small n. For example, $n \in O(n^2 - 4)$, but $n \notin O^\dagger(n^2 - 4)$, since $2 > C(2^2 - 4)$ for all $C > 0$. As another example, $n + 10 \in O(n^2)$, but $n + 10 \notin O^\dagger(n^2)$ since $0 + 10 \geq C \cdot 0^2$ for all $C > 0$.

   (c) [$\star\star\star$] $f(n) \in O^\dagger(g(n))$ if there exists $C > 0$ and a sequence of integers $n_0 \leq n_1 \leq n_2 \leq \dots$ such that $f(n_i) \leq Cg(n_i)$ for all i.

   Solution: Take f to oscillate wildly, growing faster than g at its peaks and slower at its troughs. For example, $n^2(1 + \sin(\pi n)) \in O^\dagger(1)$ since it is equal to zero whenever n is odd, but $n^2(1 + \sin(\pi n)) \notin O(1)$ since it is equal to n^2 whenever n is even. Note that you didn't have to think of the trick with $\sin(\pi n)$ — for example, by the same reasoning, it would also be valid to take $g(n) = 1$ and

   $$f(n) = \begin{cases} n^2 & \text{if } n \text{ is an even integer,} \\ 1 & \text{otherwise.} \end{cases}$$

   It's not a *nice* function, but it doesn't have to be a nice function to be a valid counterexample!

3. (a) [$\star\star\star$] Prove that if $g(n) \in \omega(1)$ and $f(n) \in o(g(n))$, then $2^{f(n)} \in o(2^{g(n)})$. (This was used in lectures as a way of dealing with unpleasant exponential functions.)

   Solution: Let $C > 0$ be arbitrary; then we must prove that $2^{f(n)} \leq C \cdot 2^{g(n)}$ whenever n is sufficiently large. Since $f(n) \in o(g(n))$, we have $f(n) \leq g(n)/2$ for all sufficiently large n; say $n \geq n_1(C)$. When this holds, we have

   $$2^{f(n)} \leq 2^{g(n)/2} = 2^{g(n)} \cdot 2^{-g(n)/2}.$$

   Since $g(n) \in \omega(1)$, when n is sufficiently large (say $n \geq n_2(C)$), we have $g(n) \geq -2\log C$. When this holds, it follows that

   $$2^{f(n)} \leq 2^{g(n)} \cdot 2^{\log C} = C \cdot 2^{g(n)}.$$

   We have therefore proved that for all $C > 0$, $2^{f(n)} \leq C \cdot 2^{g(n)}$ whenever $n \geq \max\{n_1(C), n_2(C)\}$, so we're done.

(b) [$\star\star$] What happens if $g(n) \in o(1)$?

Solution: In this case the result fails. For example, taking $f(n) = 1/n^2$ and $g(n) = 1/n$, we have $f(n) \in o(g(n))$, but $1 \leq 2^{f(n)}, 2^{g(n)} \leq 2$ for all n, so $2^{f(n)} \notin o(2^{g(n)})$.

(c) [$\star\star$] Prove that it is **not** true that if $2^{f(n)} \in o(2^{g(n)})$ then $f(n) \in o(g(n))$.

Solution: Take $f(n) = n$ and $g(n) = 2n$. Then for all $C > 0$, for all $n \geq -\log C$, we have

$$2^{f(n)} = 2^n = 2^{-n} \cdot 2^{g(n)} \leq C \cdot 2^{g(n)},$$

so $2^{f(n)} \in 2^{o(g(n))}$, but we do not have $f(n) \in o(g(n))$.

2 Induction

4. (a) [$\star\star$] Prove by induction that for all $n \geq 1$,

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$$

Solution: For the base case, taking $n = 1$, we have

$$\sum_{i=1}^{1} \frac{1}{1 \cdot 2} = \frac{1}{2} = \frac{1}{1+1}$$

as required. For the inductive step, suppose that the result holds up to $n = k$ for some $k \geq 1$. Then we express the sum up to $k+1$ in terms of the sum up to k by splitting off the last term and apply the inductive hypothesis, giving

$$\begin{aligned}\sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2},\end{aligned}$$

as required.

(b) [$\star\star$] Prove that for all $n \geq 1$,

$$\sum_{i=1}^{n} i \cdot i! = (n+1)! - 1$$

Solution: For the base case, taking $n = 1$, we have

$$\sum_{i=1}^{1} i \cdot i! = 1 \cdot 1! = 1 = (1+1)! - 1$$

as required. For the inductive step, suppose that the result holds up to some $k \geq 1$. Then we once again split off the last term of the sum and apply the inductive hypothesis, giving

$$\sum_{i=1}^{k+1} i \cdot i! = (k+1)(k+1)! + \sum_{i=1}^{k} i \cdot i! = (k+1)(k+1)! + (k+1)! - 1 = (k+2)(k+1)! - 1,$$

as required.

(c) [$\star\star$] Prove that for all $n \geq 2$,

$$\prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}$$

Solution: For the base case, taking $n = 2$, we have

$$\prod_{i=2}^{2} \left(1 - \frac{1}{i^2}\right) = 1 - \frac{1}{4} = \frac{3}{4} = \frac{2+1}{2 \cdot 2}$$

as required. For the inductive step, suppose that the result holds up to some $k \geq 2$. First, we express the product up to $k+1$ in terms of the product up to k and then apply the inductive hypothesis, yielding

$$\prod_{i=2}^{k+1} \left(1 - \frac{1}{i^2}\right) = \left(1 - \frac{1}{(k+1)^2}\right) \prod_{i=2}^{k} \left(1 - \frac{1}{i^2}\right) = \left(1 - \frac{1}{(k+1)^2}\right) \cdot \frac{k+1}{2k}. \tag{1}$$

Now, we want to show this is equal to $(k+1+1)/2k$, so let's simplify these fractions, starting with the left term:

$$1 - \frac{1}{(k+1)^2} = \frac{(k+1)^2}{(k+1)^2} - \frac{1}{(k+1)^2} = \frac{k^2 + 2k + 1 - 1}{(k+1)^2} = \frac{k(k+2)}{(k+1)^2}.$$

Combined with (1), this gives

$$\prod_{i=2}^{k+1} \left(1 - \frac{1}{i^2}\right) = \frac{k(k+2)}{(k+1)^2} \cdot \frac{k+1}{2k} = \frac{k(k+2)}{2k(k+1)} = \frac{(k+1)+1}{2(k+1)},$$

as required.

(d) [$\star\star\star$] Let $x_1 = 1$, $x_{n+1} = \sqrt{1 + 2x_n}$ for all $n \geq 1$. Prove $x_n < 4$ for all $n \geq 1$.

Solution: For the base case, taking $n = 1$, we have $x_1 = 1 < 4$ as required. For the inductive step, suppose that $x_k < 4$ for some $k \geq 1$. Then we express x_{k+1} in terms of x_k and apply the inductive hypothesis to get

$$x_{k+1} = \sqrt{1 + 2x_k} < \sqrt{1 + 2 \cdot 4} = 3 < 4,$$

as required.

(e) [$\star\star\star$] Recall from COMS10007 that the *Fibonacci sequence* is given by $F_0 = 0$, $F_1 = 1$, and

$F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Prove that writing $\phi = (1+\sqrt{5})/2$ and $\psi = (1-\sqrt{5})/2$, we have

$$F_n = \frac{\phi^n - \psi^n}{\sqrt{5}} \text{ for all } n \geq 0.$$

Prove this by induction on n.
Hint: It may help to prove that $\phi^2 = \phi + 1$ and $\psi^2 = \psi + 1$.

Solution: For the base case, taking $n = 0$ and $n = 1$, observe that

$$\begin{aligned} \frac{\phi^0 - \psi^0}{\sqrt{5}} &= \frac{1-1}{\sqrt{5}} = 0 = F_0, \\ \frac{\phi^1 - \psi^1}{2} &= \frac{1+\sqrt{5}-(1-\sqrt{5})}{4} = 1 = F_1. \end{aligned}$$

For the inductive step, suppose the result holds up to $n+1$ for some $n \geq 0$. Then expressing F_{n+2} in terms of F_{n+1} and F_n and applying the induction hypothesis, we have

$$F_{n+2} = F_{n+1} + F_n = \frac{\phi^{n+1} - \psi^{n+1} + \phi^n - \psi^n}{\sqrt{5}} = \frac{\phi^n(\phi+1) - \psi^n(\psi+1)}{\sqrt{5}}.$$

Note that we needed to apply the induction hypothesis to both F_{n+1} and F_n. (This is why we need two base cases.) The result will therefore hold if we can show $\phi + 1 = \phi^2$ and $\psi + 1 = \psi^2$; and indeed, we have

$$\begin{aligned} \phi + 1 &= \frac{3+\sqrt{5}}{2} = \frac{6+2\sqrt{5}}{4} = \phi^2, \\ \psi + 1 &= \frac{3-\sqrt{5}}{2} = \frac{6-2\sqrt{5}}{4} = \psi^2. \end{aligned}$$

(In fact, these two identities are why ϕ and ψ appear in nature so often!)

5. [⋆⋆⋆⋆] Recall from Question 4 that F_t is the t'th Fibonacci number. Prove by induction that $F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n$ for all $m, n \geq 0$.

Solution: We will prove this by induction on n, fixing an arbitrary $m \geq 0$. For the base cases with $n = 0$ and $n = 1$, we have

$$\begin{aligned} F_mF_0 + F_{m+1}F_1 &= 0F_m + 1F_{m+1} = F_{m+0+1} \\ F_mF_1 + F_{m+1}F_2 &= 1F_m + 1F_{m+1} = F_{m+1+1}, \end{aligned}$$

as required. For the inductive step, suppose the result holds up to some $n \geq 1$. Then we have

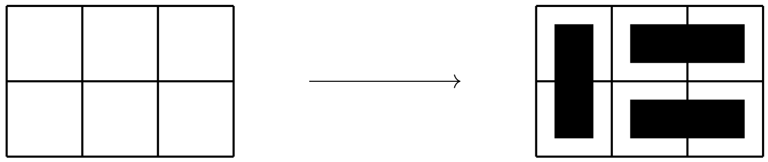
$$F_{m+(n+1)+1} = F_{m+n+2} = F_{m+n+1} + F_{m+n} = F_{m+n+1} + F_{m+(n-1)+1},$$

so on applying the induction hypothesis we obtain

$$\begin{aligned} F_{m+n+2} &= F_{m+1}F_{n+1} + F_mF_n + F_{m+1}F_n + F_mF_{n-1} \\ &= F_{m+1}(F_{n+1} + F_n) + F_m(F_n + F_{n-1}) \\ &= F_{m+1}F_{n+2} + F_mF_{n+1}, \end{aligned}$$

as required. We could also have done this by induction on m and n simultaneously, as in the lecture notes.

6. [⋆⋆⋆] Consider a rectangular board divided into an $2 \times n$ square grid for some $n \geq 1$. We can use a "domino" to cover any two horizontally or vertically adjacent squares of the board, as shown below for $n = 3$.



Prove by induction that there are exactly F_{n+1} different ways to cover every square of the board with no two dominoes overlapping, where F_{n+1} is defined as in Question 4.

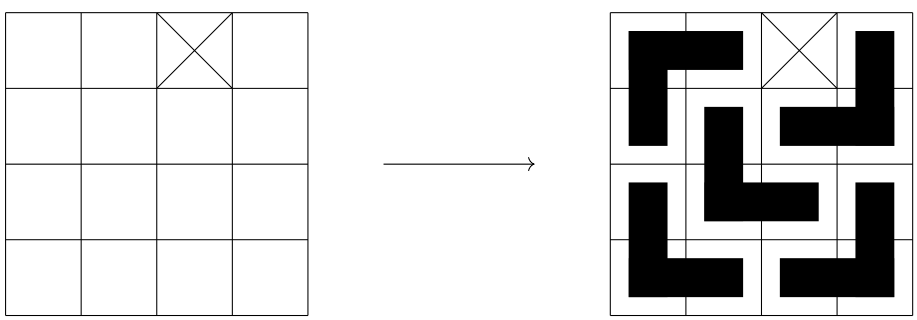
Solution: For the base case, taking $n \in \{1, 2\}$, we observe that there is only $1 = F_2$ way to cover a 2×1 board (with a vertical domino), and only $2 = F_3$ ways to cover a 2×2 board (either with two vertical dominoes or two horizontal dominoes).

For the inductive step, let $D(k)$ be the number of possible covers of a $2 \times k$ square grid. Suppose the result holds up to $n+1$ for some $n \geq 0$, and consider a $2 \times (n+2)$ square grid. In any covering, the lower-right square must be covered by either a vertical domino or a horizontal domino. If it is vertical, the remaining $2 \times (n+1)$ square grid can be covered in $D(n+1)$ ways. If it is horizontal, then the upper-right square can only be covered by another horizontal domino, and remaining $2 \times n$ grid can be covered in $D(n)$ ways. Moreover, these ways are all distinct from each other — the first $D(n+1)$ ways have a vertical domino at the right, and the last $D(n)$ ways have a horizontal domino. We have therefore shown

$$D(n+2) = D(n+1) + D(n).$$

By the induction hypothesis, it follows that $D(n+2) = F_{n+2} + F_{n+1} = F_{n+3}$, as required.

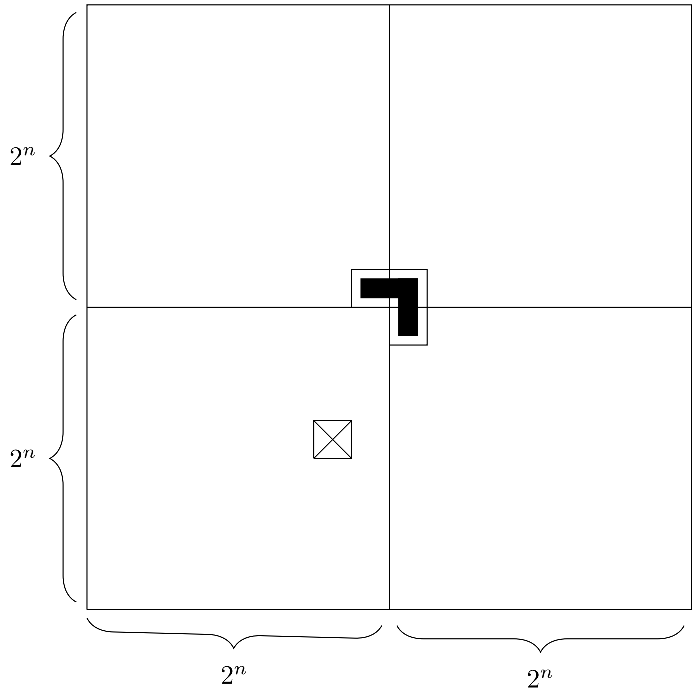
7. [⋆⋆⋆⋆⋆] Consider a rectangular board divided into an $2^n \times 2^n$ square grid for some $n \geq 1$, with one square missing. Instead of dominoes as in question 6, we wish to cover this board with non-overlapping corner-shaped trinominoes, as shown below for $n = 2$.



Prove by induction on n that this is always possible, no matter which square is missing.

Solution: For the base case, we observe that when $n = 1$, the non-missing squares of our board must be in the exact shape of a single trinomino.

For the inductive step, suppose the result holds up to n, and consider a $2^{n+1} \times 2^{n+1}$ square grid with one missing square. Observe that our board can be divided into four $2^n \times 2^n$ boards, one of which has a missing square. We add a single trinomino covering one square of each remaining board as shown below.



By the induction hypothesis, all four boards can now individually be covered by trinominoes, treating the existing trinomino as a missing square on the boards which don't already have one.