COMS20010 — Problem sheet 3

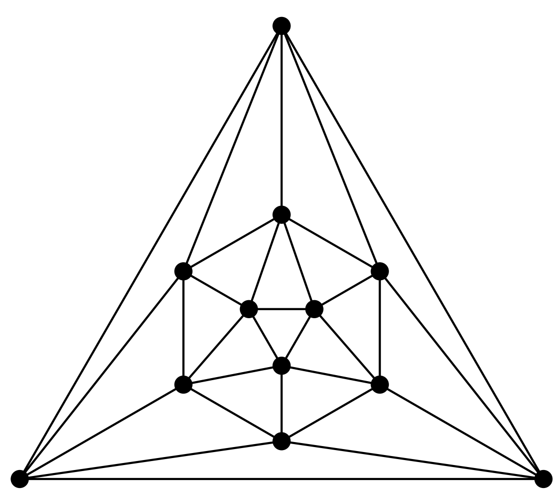
You don't have to finish the problem sheets before the class — focus on understanding the material the problem sheet is based on. If working on the problem sheet is the best way of doing that, for you, then that's what you should do, but don't be afraid to spend the time going back over the quiz and videos instead. (Then come back to the sheet when you need to revise for the exam!) I'll make solutions available shortly after the problem class. As with the Blackboard quizzes, question difficulty is denoted as follows:

- ⋆ You'll need to understand facts from the lecture notes.
- ⋆⋆ You'll need to understand and apply facts and methods from the lecture notes in unfamiliar situations.
- ⋆⋆⋆ You'll need to understand and apply facts and methods from the lecture notes and also move a bit beyond them, e.g. by seeing how to modify an algorithm.
- ⋆⋆⋆⋆ You'll need to understand and apply facts and methods from the lecture notes in a way that requires significant creativity. You should expect to spend at least 5–10 minutes thinking about the question before you see how to answer it, and maybe far longer. Only 20% of marks in the exam will be from questions set at this level.
- ⋆⋆⋆⋆⋆ These questions are harder than anything that will appear on the exam, and are intended for the strongest students to test themselves. It's worth trying them if you're interested in doing an algorithms-based project next year — whether you manage them or not, if you enjoy thinking about them then it would be a good fit.

If you think there is an error in the sheet, please post to the unit Team — if you're right then it's much better for everyone to know about the problem, and if you're wrong then there are probably other people confused about the same thing.

This problem sheet covers week 3, focusing on directed graphs, Hamilton cycles, the handshaking lemma, and trees.

1. [⋆⋆] Find a Hamilton cycle in the graph of the icosahedron, drawn below:

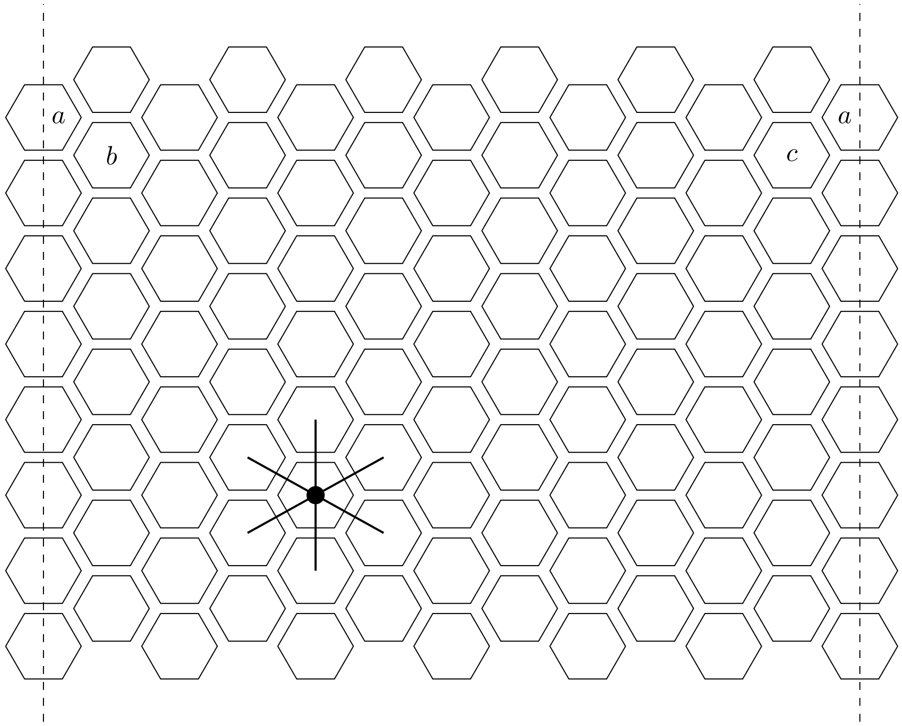


2. (a) [⋆] State the handshaking lemma for undirected graphs.

   (b) [⋆⋆] Let G be a graph with six vertices of degree five and two vertices of degree six. How many edges does G have?

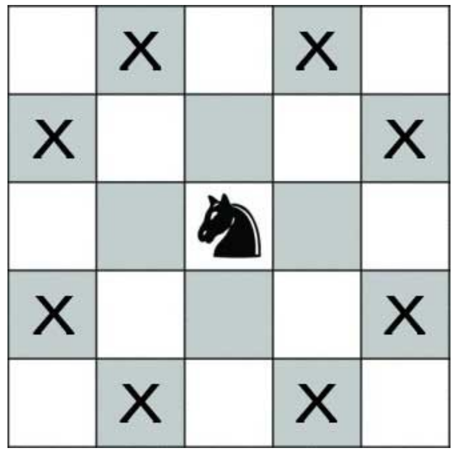
(c) [⋆⋆] In the distant future, the renowned Sol Ballet Corps is putting on a production to emphasise a spirit of love and togetherness between human colonies throughout the solar system. They have 4 dancers from Earth, 5 from Mars, 6 from the moon, and 7 from Ceres. They wish to arrange a procession in which every pair of dancers from different worlds crosses the stage exactly once. For example, one dancer from Earth will cross alongside one dancer from Mars, but not a pair of dancers both from Earth. How many pairs of dancers will need to cross the stage in total?

(d) [⋆⋆] The strategy game Civilization VI is played on a hexagonal map as shown below. The map is effectively a cylinder, with the left and right boundaries are joined together, and the top and bottom boundaries are impassable — thus in the picture below, tile a appears twice, and is adjacent to both tile b and tile c. Units can move between any pair of adjacent tiles, and this is stored in memory as a graph in which there is an edge between two tiles if they are adjacent in the map.



If the map is 90 tiles wide and 56 tiles high, how many edges does the graph have?

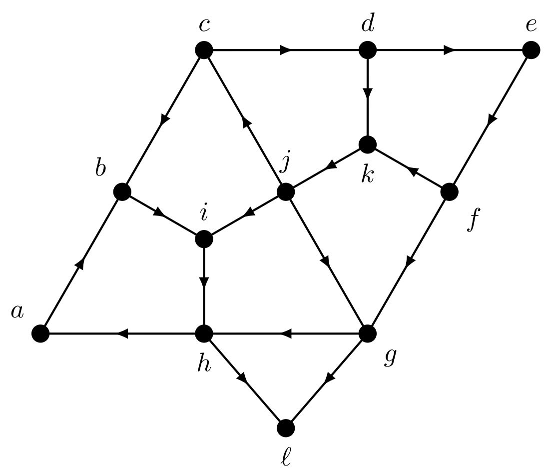
(e) [⋆⋆ but long] In lectures, we mentioned that Hamilton cycles were first studied in the context of knights' tours. The graph of an $n \times n$ knight's tour is defined as follows: the vertex set is $[n] \times [n]$, and we join (i, j) to (k, ℓ) if either $|i - k| = 1$ and $|j - \ell| = 2$ or vice versa. (In other words, viewing the vertices as squares in $n \times n$ grid, we join two squares if and only if a knight in chess can move from one to the other — see the figure below.) Prove that when $n \geq 2$, an $n \times n$ knight's tour has $4(n-1)(n-2)$ edges.



3. [$\star\star$] Let G be a graph.
   (a) Show that G is a tree if and only if G is connected, but removing any edge will disconnect it.
   (b) Show that G is a tree if and only if G has no cycles, but adding any edge will create a cycle.

4. [$\star\star\star$] Let T be a tree whose maximum degree is at least $k \geq 2$. Prove that T has at least k leaves. (This extends the result proved in lectures that any tree with at least two vertices has at least 2 leaves.)

5. Let G be the directed graph below.



   (a) [$\star\star$] What are $d^+(j)$, $d^-(j)$, $N^+(j)$ and $N^-(j)$?
   (b) [$\star\star$] Does G have an Euler walk?
   (c) [$\star$] What does it mean for a graph to be strongly connected?
   (d) [$\star\star$] Is G strongly connected? What are the strong components of G?
   (e) [$\star\star$] List all the cycles in G.
   (f) [$\star\star\star$] Show that for **any** graph G, if $X, Y \subseteq V(G)$ such that $G[X]$ is a strong component of G and $G[Y]$ is strongly connected, then either $Y \subseteq X$ or $X \cap Y = \emptyset$.
   (g) [$\star\star$] Let $\mathrm{c}(G)$ be the following directed graph, sometimes called the *condensation* of G. The vertices of $\mathrm{c}(G)$ are the strong components of G. If C_1 and C_2 are strong components of G, then there is an edge from C_1 to C_2 in $\mathrm{c}(G)$ if and only if there is an edge from some vertex of C_1 to some vertex of C_2 in G. Draw $\mathrm{c}(G)$.

(h) [$\star\star\star$] Show that for **all** directed graphs G, $\mathrm{c}(G)$ contains no cycles.

6. [$\star\star\star$] Prove that a **directed** graph $G = (V, E)$ with no isolated vertices contains an Euler walk from a vertex v to itself if and only if G is **strongly** connected and every vertex of G has equal in- and out-degrees. **Hint:** Try adapting the proof given in lectures for undirected graphs. (This is actually true for weak connectedness as well — any weakly connected digraph whose vertices have equal in- and out-degrees is strongly connected — but we won't prove this in the unit.)

7. [$\star\star\star$] In lectures we showed that when n is even, Dirac's theorem for n-vertex graphs can't be improved by lowering the degree threshold from $n/2$ to $n/2-1$. Prove that when n is odd, Dirac's theorem can't be improved by lowering the degree threshold from $n/2$ to $\lfloor n/2 \rfloor$. (**Hint:** Try considering a graph containing every possible edge between two sets of vertices.)

8. A *tournament* is a directed graph in which every pair of vertices has exactly one edge between them (in one direction) — the name comes from thinking of the graph as a competition, where if the edge between x and y is directed towards y then x beat y in their match.

   (a) [$\star\star\star\star$] Prove that any n-vertex tournament contains a vertex of out-degree at least $(n-1)/2$.

   (b) [$\star\star\star\star\star$] An *out-dominating set* in a digraph $G = (V, E)$ is a set $X \subseteq V$ such that for all vertices $v \in V \setminus X$, there is an edge **from** some vertex in X **to** v. Prove that any n-vertex tournament contains an out-dominating set of size $O(\log n)$, and give an algorithm to find it. (**Hint:** Part (a) is helpful for this.)

   (c) [$\star\star\star$] Prove that any n-vertex tournament contains an *in*-dominating set of size $O(\log n)$. (**Hint:** This can be done very quickly without rewriting your answers for (a) and (b).)

9. In lectures, we showed that we can't strengthen Dirac's theorem by lowering the minimum degree bound. This question is about strengthening Dirac's theorem in another direction. Let G be an n-vertex graph with $n \geq 3$. For all pairs $\{u, v\}$, we write $G + \{u, v\}$ for the graph formed from G by adding $\{u, v\}$ as an edge, i.e. $G + \{u, v\} = (V(G), E(G) \cup \{\{u, v\}\})$.

   (a) [$\star\star\star\star$] Show that if $d(u) + d(v) \geq n$, then G contains a Hamilton cycle if and only if $G + \{u, v\}$ contains a Hamilton cycle. (**Hint:** You will need to use an idea from the proof of Dirac's theorem in lectures.)

   (b) [$\star\star\star$] Show that if $d(u) + d(v) \geq n$ for all non-adjacent $u, v \in V(G)$, then G contains a Hamilton cycle;