Proof by induction (recap)
COMS20010 2020, Video lecture 1-2

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Write $S(n)$ for the statement that $\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}$ for all $x \neq 1$.
So for example, $S(0)$ says that $x^0 = \frac{x^{0+1}-1}{x-1} = 1$ for all x, which is true.

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Rather than proving $S(n)$ for all n, we write an algorithm which, given n as an input, outputs a proof of $S(n)$. So since we can prove $S(n)$ for all n, it must hold for all n!

Let $S(n)$ be the statement that $\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}$ for all $x \neq 1$.
We want to prove by induction that $S(n)$ holds for all n.

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The "program" has two subroutines:

- `BaseCase()` outputs a proof of $S(0)$;
- `InductiveStep(`k`)` outputs a proof of $S(k+1)$ from $S(0), \ldots, S(k)$.

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And the pseudocode reads:

Input: An integer $n \geq 0$.
Output: A proof of $S(n)$.

begin
  Output `BaseCase()`.
  foreach k *in* $\{0, \ldots, n-1\}$ **do**
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  Halt.

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But this program is the same for every induction proof, so we only have to specify `BaseCase` and `InductiveStep`.

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We have

$$\sum_{i=0}^{k+1} x^i = \sum_{i=0}^{k} x^i + x^{k+1}$$

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And so we're done! □

Strong induction

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For example, this proof of the inductive step is also valid:

$$\sum_{i=0}^{k+1} x^i = \sum_{i=0}^{\lfloor k/2 \rfloor} x^i + \sum_{i=\lfloor k/2 \rfloor + 1}^{k+1} x^i$$

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This technique is sometimes called **strong induction**.

Another example

Consider the following (awful) pseudocode for a function `Increment`(y):

Input: An integer $y > 0$.
Output: $y + 1$.

begin
    if $y = 0$ **then**
        Return 1.
    else if $y \pmod 2 = 0$ **then**
        Return $y + 1$.
    else
        Return $2 \cdot \texttt{Increment}(\lfloor y/2 \rfloor)$.

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Base case: If $y = 0$, we return $1 = y + 1$. ✓

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Writing $y = 2z+1$, we have $\lfloor y/2 \rfloor = z$, so this is $2(z+1) = y+1$. □

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Input: Integers $m, n \geq 0$.
Output: A proof of $S(m, n)$.

```
begin
    if m = 0 or n = 0 then
        Output BaseCase().
    else
        Output Proof(m − 1, n − 1).
        Output InductiveStep(m, n).
    Halt.
```

Other induction schemes

This corresponds to an induction proof of:

- **Base case:** Prove $S(m, 0)$ for all m and $S(0, n)$ for all n;
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This is valid! There are lots of other induction schemes, e.g.:

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  - $S(m, n-1)$ when $m = 0$ and $n \geq 1$;
  - $S(m-1, n)$ when $m \geq 1$ and $n = 0$.
- For a one-variable statement $S(n)$: The base cases are $S(0)$ and $S(1)$, and the inductive step proves $S(n)$ from $S(n-2)$ for all $n \geq 2$.

Other induction schemes

Without getting into foundational stuff: if you can put your induction proof in the form of an "induction program" that will output a (finite!) proof for any parameter choice, then you have a valid proof by induction.

This is useful in dealing with functions on multiple variables, or complicated structures that can be broken down into simpler parts.

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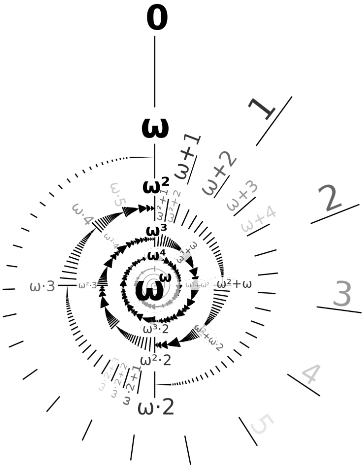
But beware subtle errors! (See the quiz...)

Non-examinable: transfinite induction



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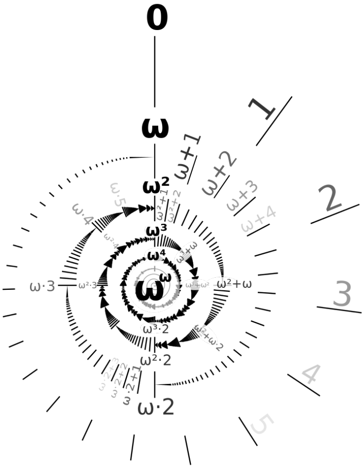

Just say no.