Defining O-notation (recap)
COMS20010 2020, Video lecture 1-3

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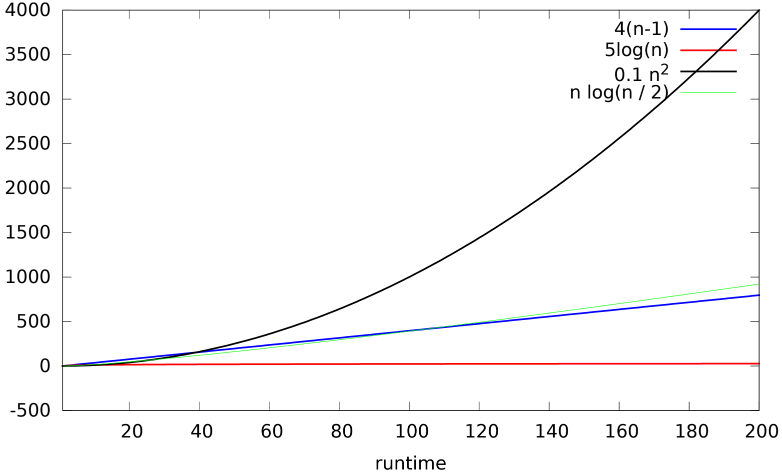
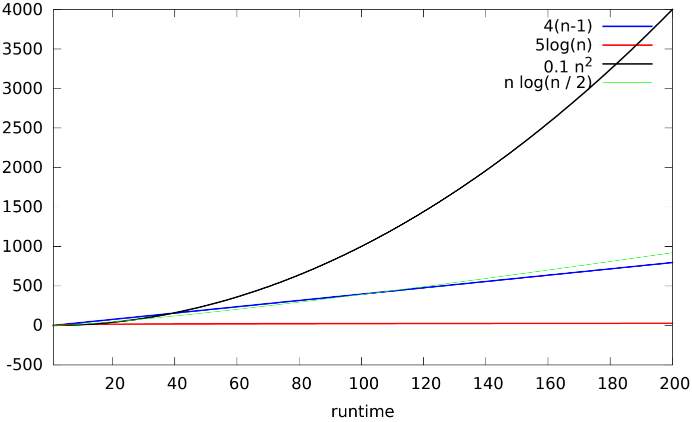
Why O-notation?

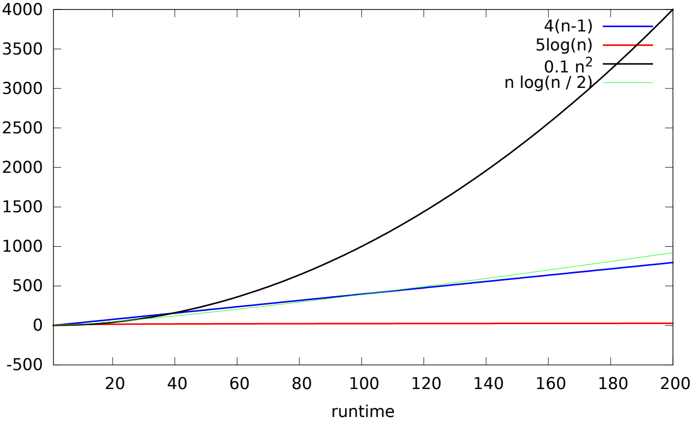
Intuition: As input sizes get large, asymptotic growth rate matters more than constant factors. Also, constant factors are implementation-dependent. So we focus on growth rate.





We would like $f(n) \in O(g(n))$ to mean:

- $f(n)$ "grows no faster than" $g(n)$, ignoring constant factors.



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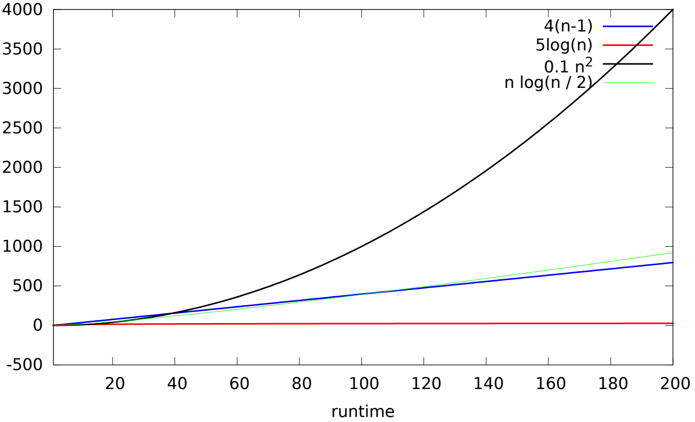
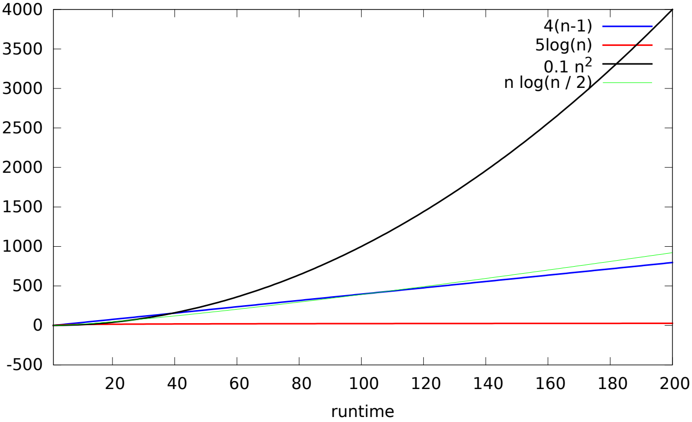
- $f(n)$ "grows no faster than" $g(n)$, **ignoring constant factors**.
- There exists $C > 0$ such that $f(n)$ "grows no faster than" $C \cdot g(n)$.



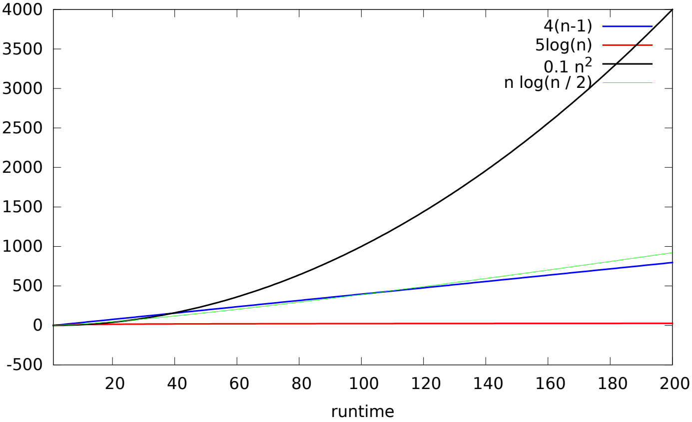
We would like $f(n) \in O(g(n))$ to mean:

- There exists $C > 0$ such that $f(n)$ **"grows no faster than"** $C \cdot g(n)$.
- There exists $C > 0$ such that $f(n) \leq C \cdot g(n)$ whenever n is sufficiently large.



We would like $f(n) \in O(g(n))$ to mean:

- There exists $C > 0$ such that $f(n) \leq C \cdot g(n)$ whenever n is **sufficiently large**.
- There exist C, $n_0 > 0$ such that $f(n) \leq C \cdot g(n)$ whenever $n \geq n_0$.



We would like $f(n) \in O(g(n))$ to mean:

There exist $C, n_0 > 0$ such that $f(n) \leq C \cdot g(n)$ whenever $n \geq n_0$. ✓

This rigorous definition is "just" a more precise version of our intuition.

Other O-notation

$f(n) \in O(g(n))$ is good notation for "f grows no faster than g, ignoring constants". But what if we want to say "g grows no slower than f"?

Notation	Intuitive meaning	Analogue
$f(n) \in O(g(n))$	f grows at most as fast as g	$\leq$
$f(n) \in \Omega(g(n))$	f grows at least as fast as g	$\geq$
$f(n) \in \Theta(g(n))$	f at the same rate as g	$=$
$f(n) \in o(g(n))$	f grows strictly less fast than g	$<$
$f(n) \in \omega(g(n))$	f grows strictly faster than g	$>$

Notation	Formal definition
$f(n) \in O(g(n))$	$\exists C,\ n_0 \colon \forall n \geq n_0 \colon f(n) \leq C \cdot g(n)$
$f(n) \in \Omega(g(n))$	$\exists c,\ n_0 \colon \forall n \geq n_0 \colon f(n) \geq c \cdot g(n)$
$f(n) \in \Theta(g(n))$	$\exists c,\ C,\ n_0 \colon \forall n \geq n_0 \colon c \cdot g(n) \leq f(n) \leq C \cdot g(n)$
$f(n) \in o(g(n))$	$\forall C \colon \exists n_0 \colon \forall n \geq n_0 \colon f(n) \leq C \cdot g(n)$
$f(n) \in \omega(g(n))$	$\forall c \colon \exists n_0 \colon \forall n \geq n_0 \colon f(n) \geq c \cdot g(n)$

Examples

Example 1: Prove $n^2 - 5n + 12 \in \Theta(n^2)$ directly from the definition.

Remember the definition: proving $n^2 - 5n + 12 \in \Theta(n^2)$ **means** proving there exist c, C and n_0 such that $cn^2 \leq n^2 - 5n + 12 \leq Cn^2$ for all $n \geq n_0$.

We expect $n^2 - 5n + 12 \approx n^2$ for large n, so we could e.g. set $c = 1/2$ and $C = 2$ and solve the quadratic. But let's be lazy! No need to optimise. We have

$$n^2 - 5n + 12 \leq n^2 + 12 = n^2(1 + \tfrac{12}{n^2}),$$
$$n^2 - 5n + 12 \geq n^2 - 5n = n^2(1 - \tfrac{5}{n}).$$

Looking at it like this, it's much easier to see that

$$n^2 - 5n + 12 \leq 13n^2 \text{ for all } n \geq 1,$$
$$n^2 - 5n + 12 \geq n^2/2 \text{ for all } n \geq 10 \text{ (so } \tfrac{5}{n} \leq \tfrac{1}{2}\text{)}.$$

So we prove $n^2 - 5n + 12 \in \Theta(n^2)$ by taking $c = \frac{1}{2}$, $C = 13$, and $n_0 = 10$.

Examples

Example 2: Prove $n! \in \omega(2^n)$ directly from the definition.

Remember the definition: proving $n! \in \omega(2^n)$ **means** proving that for all $c > 0$, there exists n_0 such that for all $n \geq n_0$, $n! \geq c \cdot 2^n$.

So we're given a constant c, and we need to show $n! \geq c \cdot 2^n$ when n is sufficiently large. Remember we have

$$n! = \underbrace{n \cdot (n-1) \cdot \dots \cdot 1}_{n \text{ terms}}, \qquad 2^n = \underbrace{2 \cdot 2 \cdot \dots \cdot 2}_{n \text{ terms}}.$$

So we have a lot of wiggle room to bound things term-by-term.

Let's use the fact that $n! \geq 4^{n-3} = 2^n \cdot 2^{n-6}$.
Thus $n! \geq c \cdot 2^n$ whenever $2^{n-6} \geq c$, i.e. whenever $n \geq \log c + 6$.

So we prove $n! = \omega(2^n)$ by taking $n_0 \geq \log c + 6$.

Multi-variable O-notation

We will often need O-notation for functions of more than one variable.

For example, an algorithm running on an n-vertex m-edge graph will often have running time depending on both m and n.

What does it mean to say that e.g. $f(m,n) \in O(mn)$ or $f(m,n) \in \Theta(m^2 \log n)$?

The only difference is that instead of requiring n to be sufficiently large, we require **all** variables to be sufficiently large.

For example, $f(m,n) \in O(g(m,n))$ when there exist C, $\boldsymbol{m_0}$ **and** $\boldsymbol{n_0}$ such that $f(m,n) \le C \cdot g(m,n)$ whenever $m \ge m_0$ **and** $n \ge n_0$.

All the useful properties of single-variable O-notation (see next video!) carry over to multi-variable O-notation, so e.g. if $f(m,n) \in O(g(m,n))$ and $f(m,n) \in \Omega(g(m,n))$ then we still have $f(m,n) \in \Theta(g(m,n))$.

An important clarification (added after recording)

O-notation can behave strangely with negative functions.

But we only care about O-notation for running times, which are positive!

So whenever you are asked to prove something general about O-notation in this course, you can assume the functions involved are non-negative.

But logarithms get used to bound running times all the time, and e.g. $n\log(n/100)$ is negative for small n. Since it's positive for large n, we'd still like to be able to say e.g. $n\log(n/100) \in \Theta(n\log n)$.

So the formal requirement is that the functions involved are **eventually non-negative** — that is, before we can say $f(n) \in O(g(n))$ or similar, we require that $f(n), g(n) \geq 0$ for all sufficiently large n.

Any fact that holds about O-notation for non-negative functions will also hold for eventually non-negative functions, by taking n_0 large enough that "eventually non-negative" becomes "non-negative".