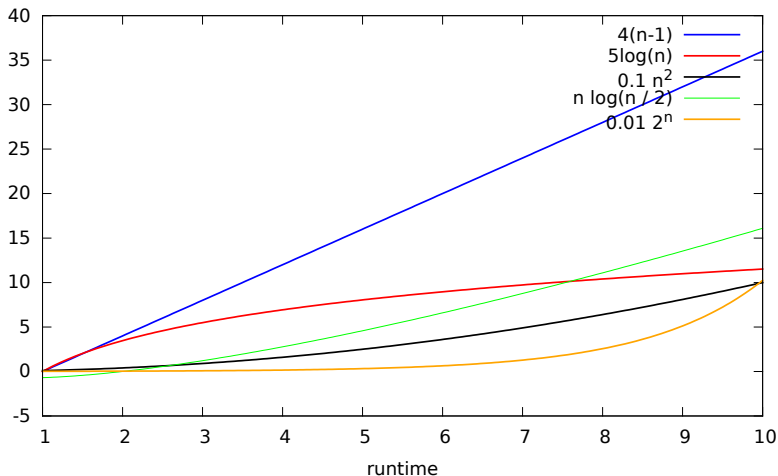


Defining O-notation (recap)
COMS20010 2020, Video lecture 1-3

John Lapinskas, University of Bristol

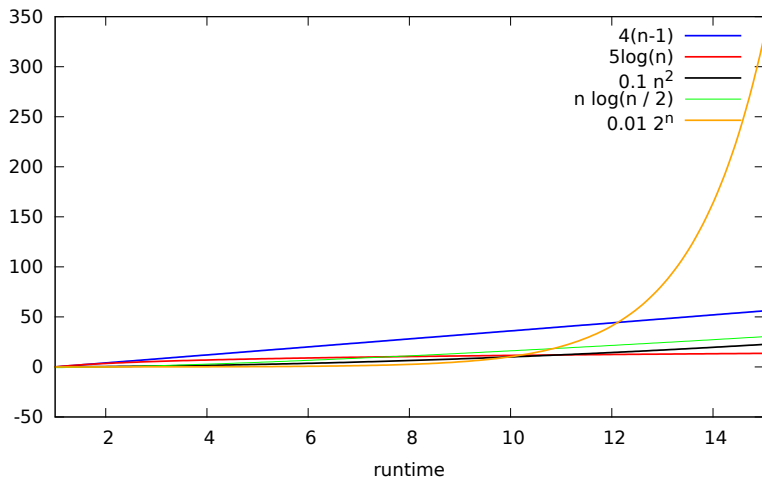
Why O-notation?

Intuition: As input sizes get large, asymptotic growth rate matters more than constant factors. Also, constant factors are implementation-dependent. So we focus on growth rate.



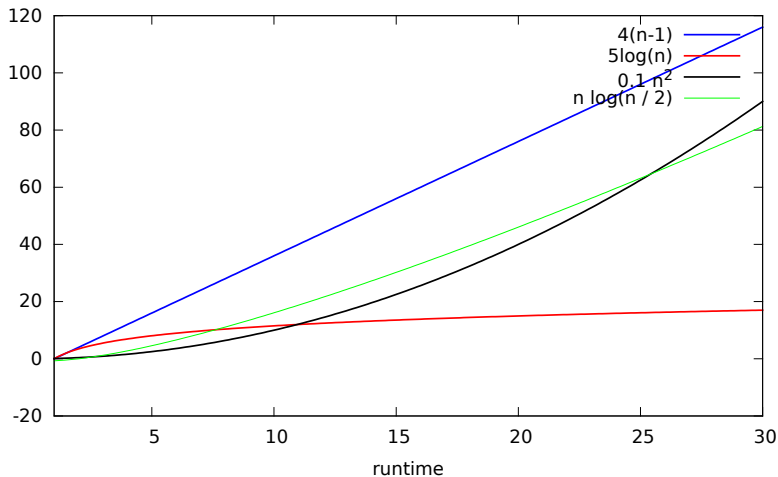
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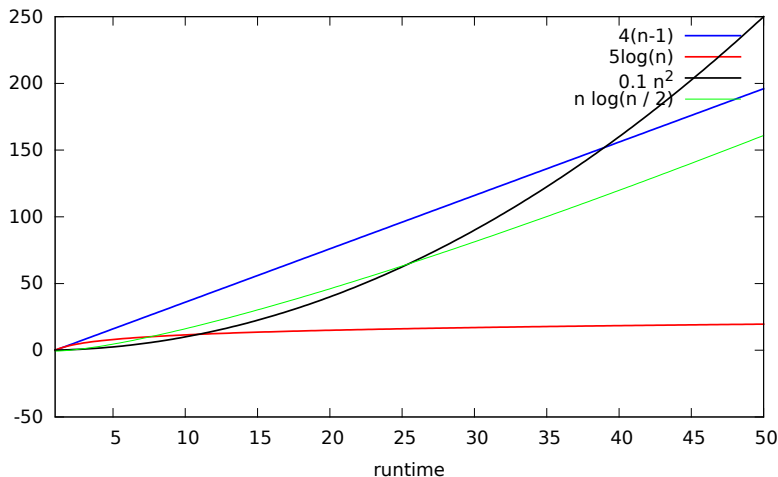
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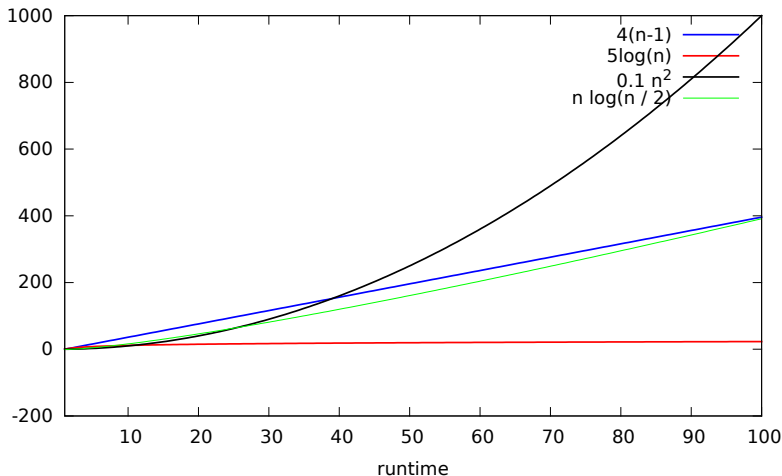
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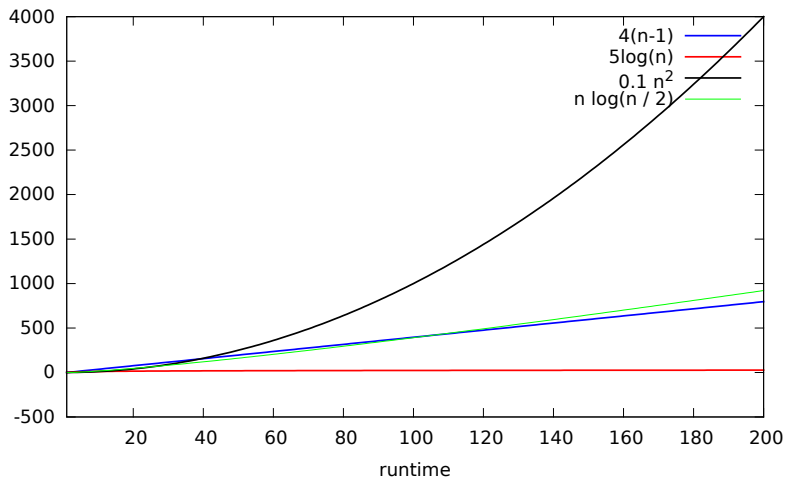
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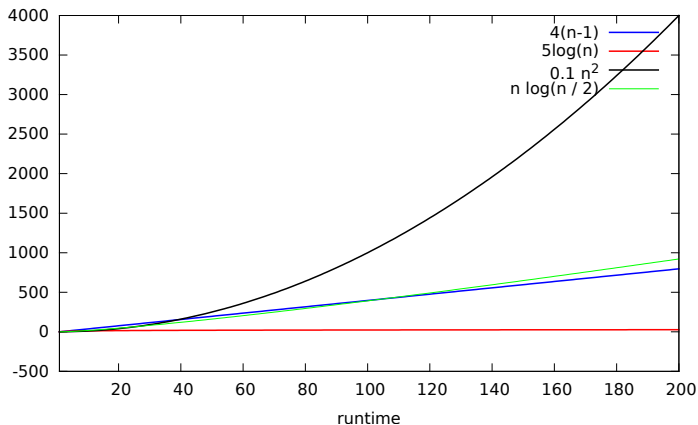
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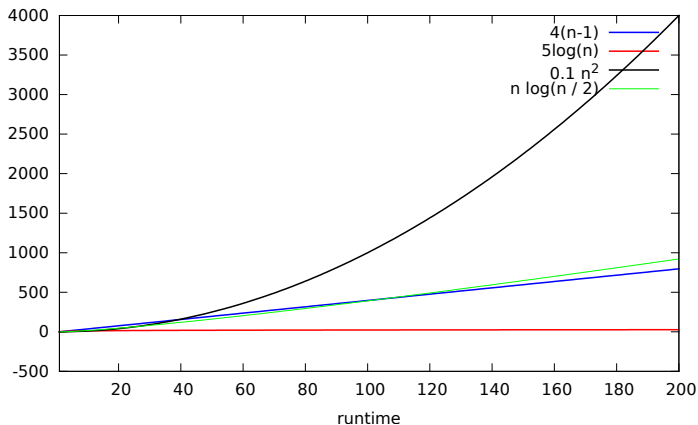
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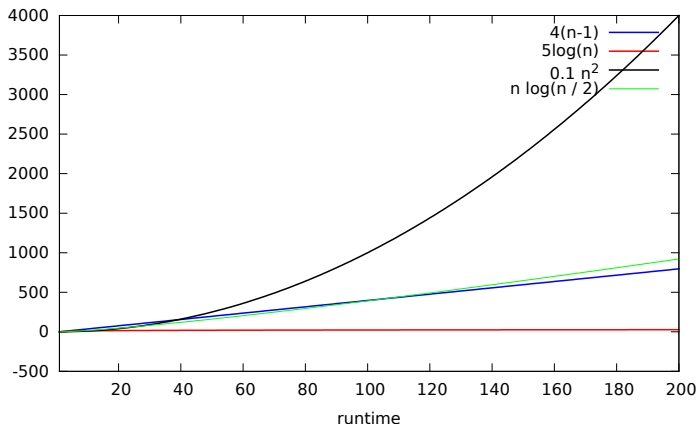
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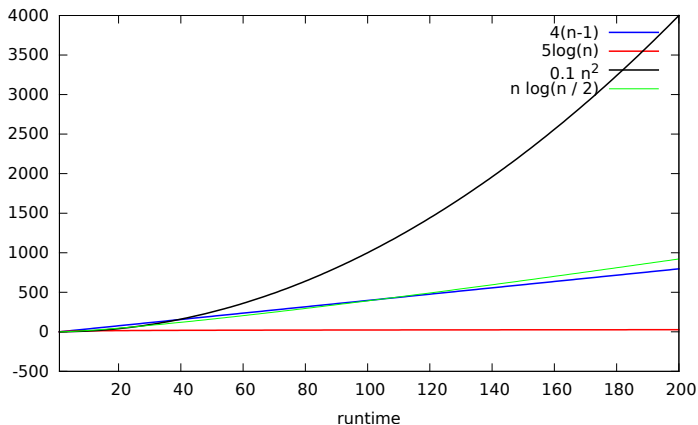
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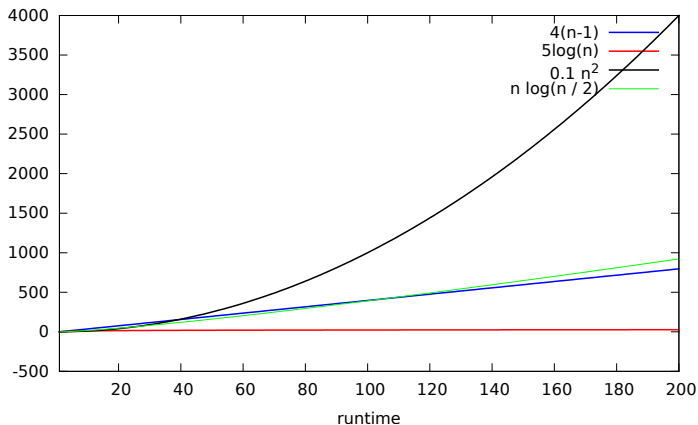
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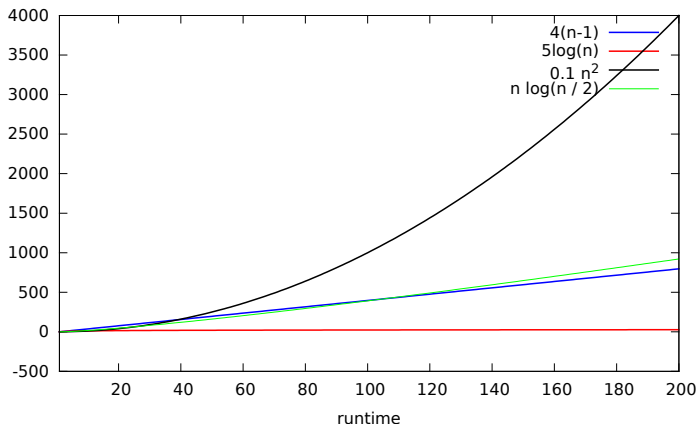
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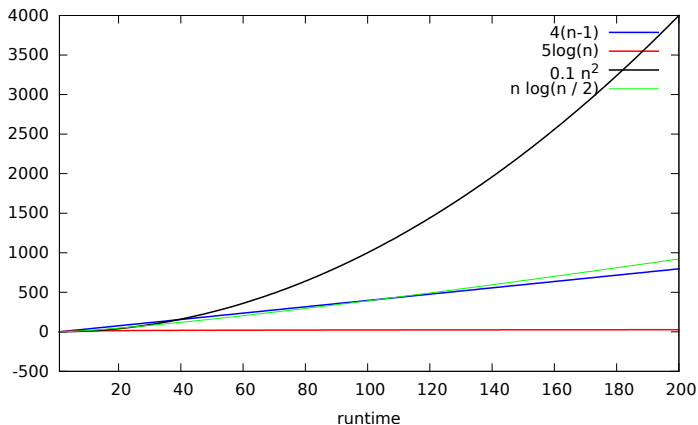
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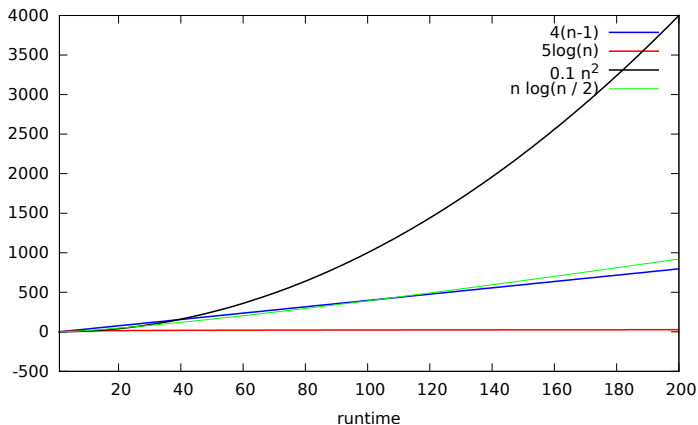
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- There exists $C > 0$ such that $f(n)$ **“grows no faster than”** $C \cdot g(n)$.
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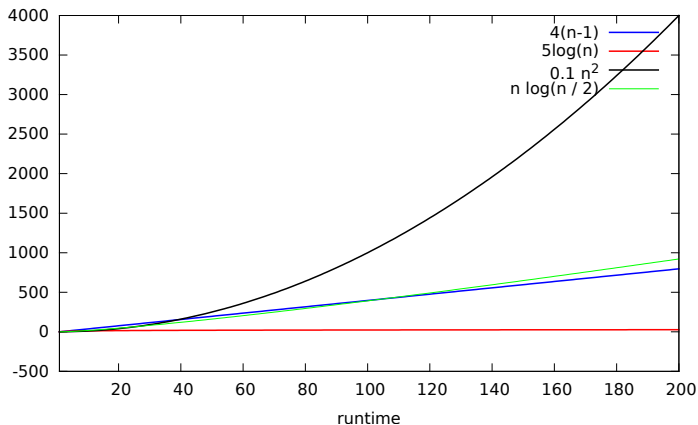
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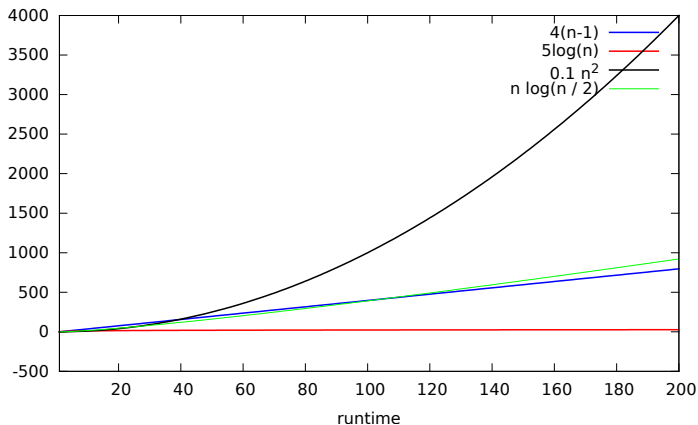
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This rigorous definition is “just” a more precise version of our intuition.

Other O-notation

$f(n) \in O(g(n))$ is good notation for “ f grows no faster than g , ignoring constants”. But what if we want to say “ g grows no slower than f ”?

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$f(n) \in \Omega(g(n))$	f grows at least as fast as g	\geq
$f(n) \in \Theta(g(n))$	f at the same rate as g	$=$
$f(n) \in o(g(n))$	f grows strictly less fast than g	$<$
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$$n^2 - 5n + 12 \leq n^2 + 12 = n^2\left(1 + \frac{12}{n^2}\right),$$

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So we prove $n^2 - 5n + 12 \in \Theta(n^2)$ by taking $c = \frac{1}{2}$, $C = 13$, and $n_0 = 10$.

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So we're given a constant c , and we need to show $n! \geq c \cdot 2^n$ when n is sufficiently large. Remember we have

$$n! = \underbrace{n \cdot (n-1) \cdot \dots \cdot 1}_{n \text{ terms}}, \quad 2^n = \underbrace{2 \cdot 2 \cdot \dots \cdot 2}_{n \text{ terms}}.$$

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So we prove $n! = \omega(2^n)$ by taking $n_0 \geq \log c + 6$.

Multi-variable O-notation

We will often need O-notation for functions of more than one variable.

For example, an algorithm running on an n -vertex m -edge graph will often have running time depending on both m and n .

What does it mean to say that e.g. $f(m, n) \in O(mn)$ or $f(m, n) \in \Theta(m^2 \log n)$?

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The only difference is that instead of requiring n to be sufficiently large, we require **all** variables to be sufficiently large.

For example, $f(m, n) \in O(g(m, n))$ when there exist C , m_0 and n_0 such that $f(m, n) \leq C \cdot g(m, n)$ whenever $m \geq m_0$ **and** $n \geq n_0$.

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All the useful properties of single-variable O-notation (see next video!) carry over to multi-variable O-notation, so e.g. if $f(m, n) \in O(g(m, n))$ and $f(m, n) \in \Omega(g(m, n))$ then we still have $f(m, n) \in \Theta(g(m, n))$.

An important clarification (added after recording)

O-notation can behave strangely with negative functions.

But we only care about O-notation for running times, which are positive!

So whenever you are asked to prove something general about O-notation in this course, you can assume the functions involved are non-negative.

But logarithms get used to bound running times all the time, and e.g. $n \log(n/100)$ is negative for small n . Since it's positive for large n , we'd still like to be able to say e.g. $n \log(n/100) \in \Theta(n \log n)$.

So the formal requirement is that the functions involved are **eventually non-negative** — that is, before we can say $f(n) \in O(g(n))$ or similar, we require that $f(n), g(n) \geq 0$ for all sufficiently large n .

Any fact that holds about O-notation for non-negative functions will also hold for eventually non-negative functions, by taking n_0 large enough that “eventually non-negative” becomes “non-negative”.