Weighted interval scheduling
COMS20010 (Algorithms II)

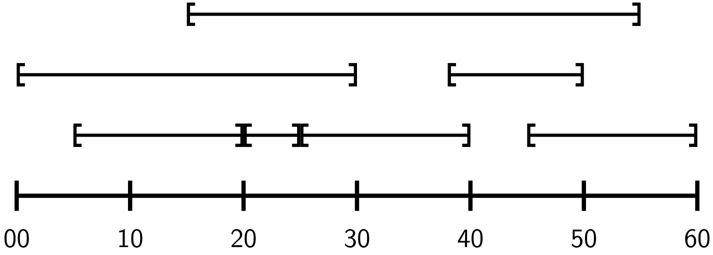
John Lapinskas, University of Bristol

Unweighted interval scheduling (recap from week 2)

Motivation: A satellite imaging service wants to use its camera to fill as many orders as possible, but it can only take one picture at once.

Input: A set of intervals, e.g.
$\mathcal{R} = \{(0, 30), (5, 20), (15, 55), (20, 25), (25, 40), (38, 50), (45, 60)\}$.

Output: A maximum **compatible** set $\mathcal{R}' \subseteq \mathcal{R}$ — that is, for all $(s, f), (s', f') \in \mathcal{R}'$, we have either $s', f' \geq f$ or $s', f' \leq s$.



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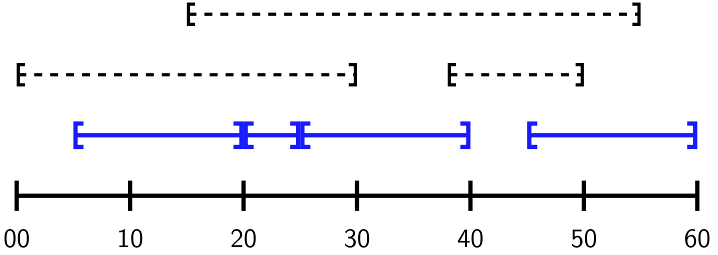

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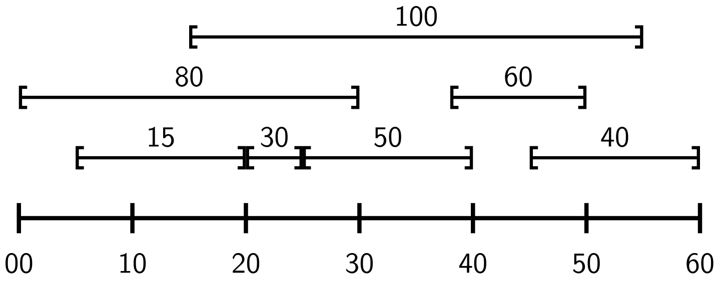

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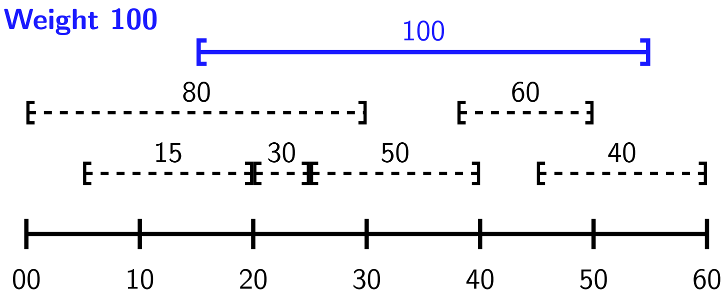

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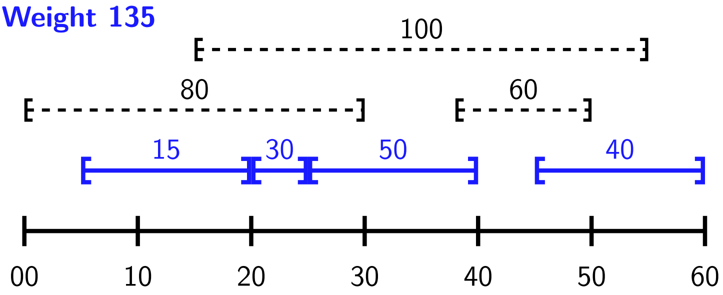

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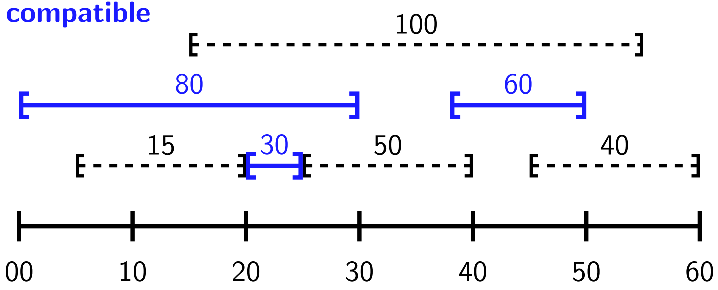

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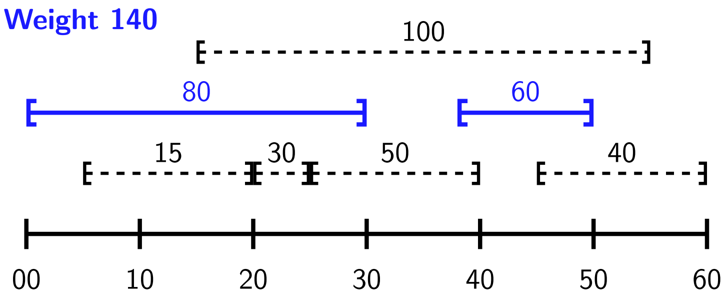

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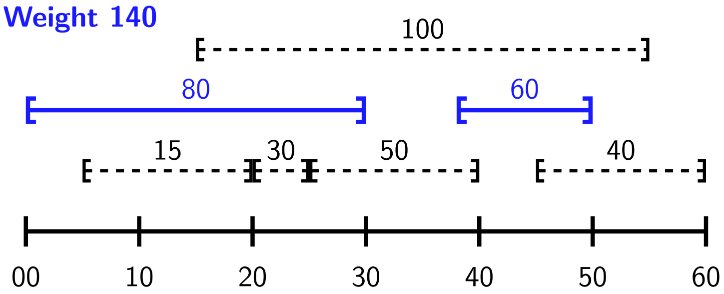

In this case, the desired output has weight 140. But our old greedy algorithm fails! There is **no** known greedy algorithm for this problem.

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Why call it "dynamic programming"?

Because Richard Bellman needed to sell it to an idiot politician and "dynamic" was a fashionable word! If it had been invented today, it would have been called "agile blockchain programming in the cloud"...

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Input: A set of intervals $\mathcal{R}$ and a **weight function** $w\colon \mathcal{R} \to \mathbb{Q}_{\geq 0}$.
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You can think of weighted interval scheduling as a sequence of **choices**: Do I include this interval in my output, or not?

Our greedy algorithm decided "yes" or "no" based only on finishing times. But to reduce a problem to itself, we consider the **effect** of each choice.

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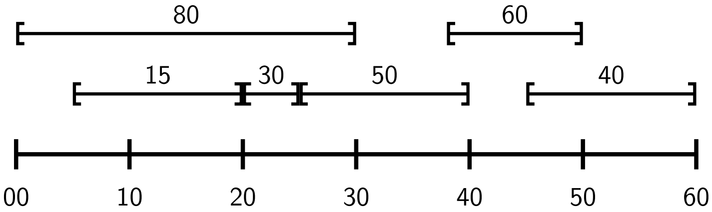

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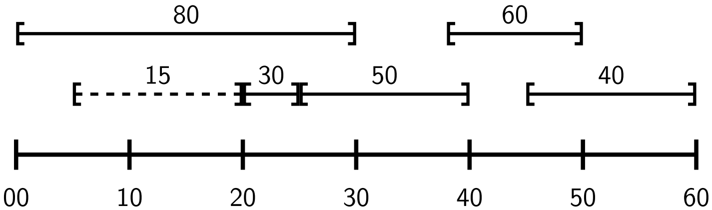

If we don't include some interval I: Then the maximum-weight compatible set will be the same as $\mathrm{WIS}(\mathcal{R} \setminus \{I\})$.

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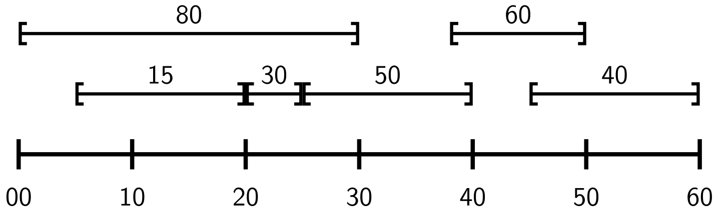

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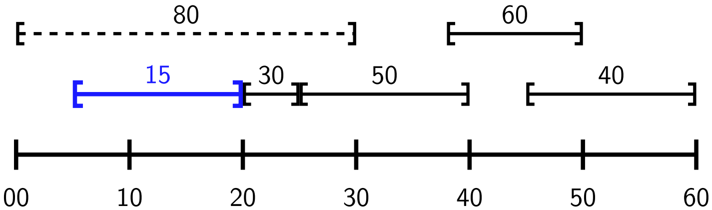

If we do include some interval I: Then we can't include any interval in the set $X_I \subseteq \mathcal{R}$ of intervals intersecting I, or we lose compatibility. But the maximum-weight compatible set will be I together with $\mathrm{WIS}(\mathcal{R} \setminus X_I)$.

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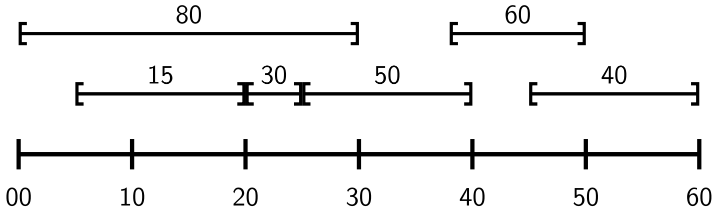

So overall, the highest-weight compatible set will be either $\mathrm{WIS}(\mathcal{R} \setminus \{I\})$ or $\{I\} \cup \mathrm{WIS}(\mathcal{R} \setminus X_I)$, whichever has higher weight.

(See problem sheet 8 question 4 for more examples!)

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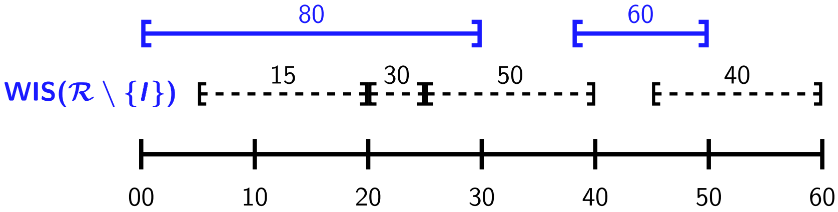

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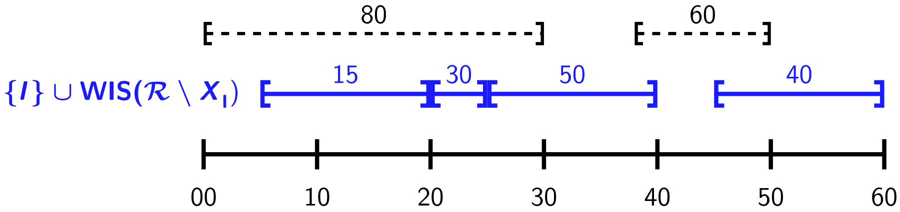

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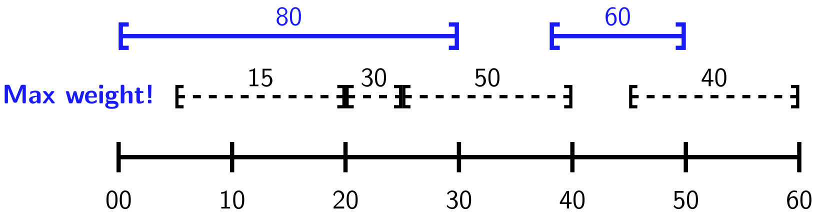

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The full recursive algorithm

Algorithm: WIS

Input : An array $\mathcal{R}$ of n requests and a weight function w.
Output : A maximum-weight compatible subset of $\mathcal{R}$.

begin
  if $\mathcal{R} = \emptyset$ **then**
    Return $\emptyset$.
  else
    Choose $I \in \mathcal{R}$ arbitrarily.
    Find the set X_I of intervals in $\mathcal{R}$ incompatible with I.
    $S_{\text{out}} \leftarrow \text{WIS}(\mathcal{R} \setminus \{I\}, w)$.
    $S_{\text{in}} \leftarrow \{I\} \cup \text{WIS}(\mathcal{R} \setminus X_I, w)$.
    if $w(S_{\text{out}}) > w(S_{\text{in}})$ **then**
      Return S_{out}.
    else
      Return S_{in}.

Note that this algorithm will work **regardless** of how we pick each I.

Next video, we will exploit this to make the algorithm run much faster...