When does a graph have an Euler walk?
COMS20010 (Algorithms II)

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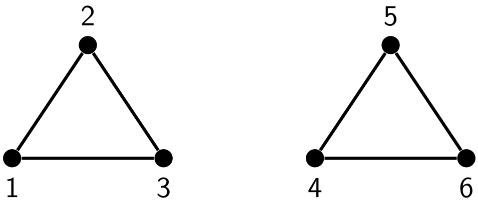
Last video: If G has an Euler walk, then either:

- every vertex of G has even degree; or
- all but two vertices v_0 and v_k have even degree, and any Euler walk must have v_0 and v_k as endpoints.

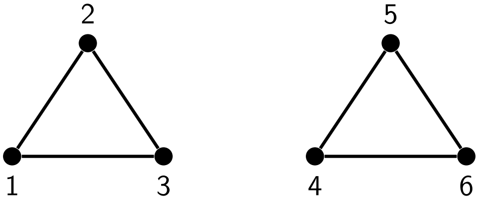
Does every graph satisfying one of these have an Euler walk? No! E.g.:



Every vertex has even degree, but we can't cross between the triangles. We need some more definitions to rule this case out...

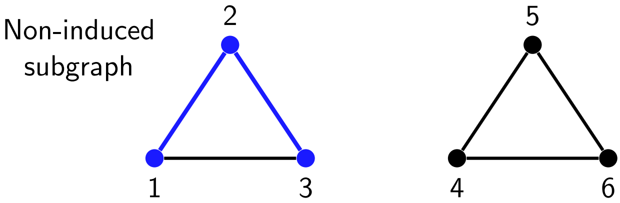
Connectedness

A **path** is a walk in which no vertices repeat.
A graph is **connected** if any two vertices are joined by a path. So...



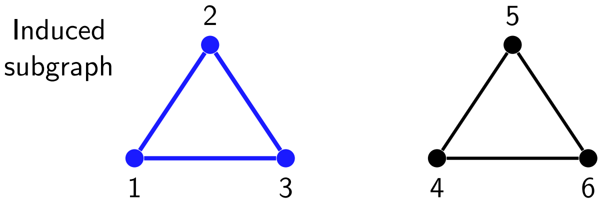
This graph is **not** connected because there's no path from 3 to 4 (say).

Exercise: Two vertices are joined by a path if and only if they are joined by a walk. (Paths are just more convenient to use.)



We'd also like to have names for the left and right triangles.

Let $G = (V, E)$ be a graph.
A **subgraph** $H = (V_H, E_H)$ of G is a graph with $V_H \subseteq V$ and $E_H \subseteq E$.
H is an **induced subgraph** if $V_H \subseteq V$ and $E_H = \{e \in E \colon e \subseteq V_H\}$.



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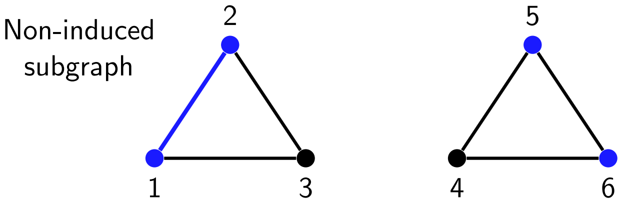

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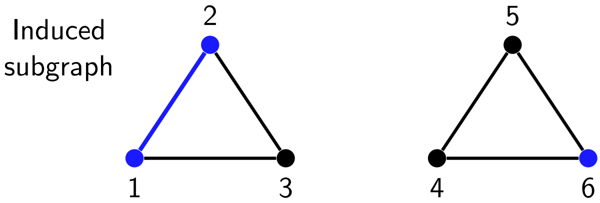

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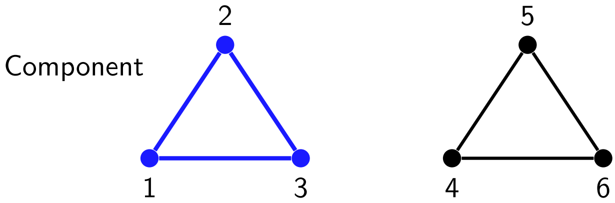
For all vertex sets $X \subseteq V$, the graph **induced** by X is

$$G[X] = (X, \{e \in E \colon e \subseteq X\}).$$

So if G is the above graph, then the subgraph shown is $G[\{1, 2, 6\}]$.



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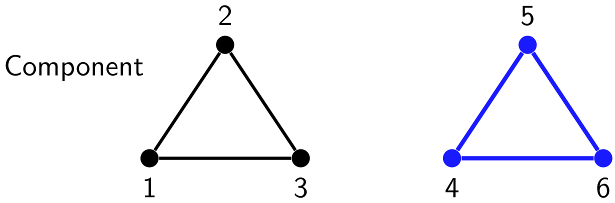
A **component** H of G is a maximal connected induced subgraph of G. So $H = G[V_H]$ is connected, but $G[V_H \cup \{v\}]$ is disconnected for all $v \in V \setminus V_H$.

Here, the two components of G are the left and right triangles.
A connected graph only has one component, namely the graph itself.



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A connected graph only has one component, namely the graph itself.

We call a single-vertex component an **isolated vertex**.

A **path** is a walk in which no vertices repeat.
A graph is **connected** if any two vertices are joined by a path.

Theorem: Let $G = (V, E)$ be a **connected** graph, and let $u, v \in V$. Then G has an Euler walk from u to v if and only if either:

(i) $u = v$ and every vertex of G has even degree; or

(ii) $u \neq v$ and every vertex of G has even degree except u and v.

We have already proved the "only if" direction. ✓

Proof of "if", case (i):
Suppose G is connected, $v \in V$, and every vertex of G has even degree.

Idea: Try working greedily!

Form a walk $W = w_0 \dots w_k$ as follows:

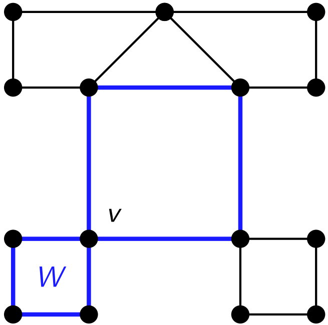
- take $w_0 = v$;
- take w_i to be an arbitrary neighbour of w_{i-1} such that $\{w_{i-1}, w_i\}$ is not already a W-edge;
- stop when every edge out of w_i is already a W-edge.

A **path** is a walk in which no vertices repeat.
A graph is **connected** if any two vertices are joined by a path.

Goal: Let $G = (V, E)$ be a **connected** graph with even vertex degrees. Then for all $v \in V$, G has an Euler walk from v to v.

Idea: Form a walk $W = w_0 \dots w_k$ by taking $w_0 = v$, extending greedily without reusing edges, and stopping when $N(w_i) \subseteq \{w_0, \dots, w_{i-1}\}$.

This need not give an Euler walk...



But it still gives us something.

Claim: $w_k = w_0 = v$.

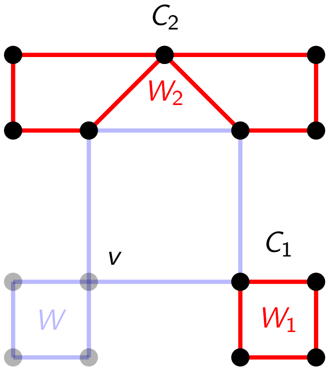
Proof: For a vertex x, how many W-edges are incident to x?

- Two for each time x appears in $\{w_1, \dots, w_{k-1}\}$;
- Plus one if $x = w_0$;
- Plus one if $x = w_k$.

We know w_k has $d(w_k)$ W-edges, and $d(w_k)$ is even, so $w_0 = w_k$. ✓

Goal: Let $G = (V, E)$ be a **connected** graph with even vertex degrees. Then for all $v \in V$, G has an Euler walk from v to v.

Lemma: G has a non-trivial walk W from v to v with no reused edges. ✓



Idea: Strong induction on $|E|$.

Base case: $|E| = 0$, immediate.

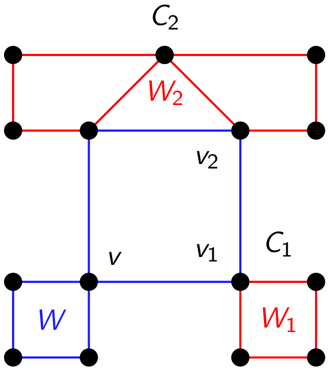
Induction step: Apply induction hypothesis to find Euler walks W_i for all non-empty components C_i of the subgraph $G - W$ formed by removing W's edges from G:

- Each vertex has even degree in both W and G, and hence also in $G - W$. ✓
- Each component C_i is connected by definition. ✓

Goal: Let $G = (V, E)$ be a **connected** graph with even vertex degrees. Then for all $v \in V$, G has an Euler walk from v to v.

Lemma: G has a non-trivial walk W from v to v with no reused edges. ✓

Induction hypothesis $\Rightarrow$ each non-trivial component C_i of $G - W$ has an Euler walk W_i from any vertex to itself. ✓



Idea: "Walk along W until we hit C_1, then follow W_1, then go back to W, and so on."

G connected $\Rightarrow$ there is a path P_i from v to each C_i. Let v_i be the first vertex in C_i on P_i.

Then some edge incident to v_i must have been removed in $G - W$, or the vertex before v_i on P_i would have been part of C_i.

So $v_i \in W$.

Goal: Let $G = (V, E)$ be a **connected** graph with even vertex degrees. Then for all $v \in V$, G has an Euler walk from v to v.

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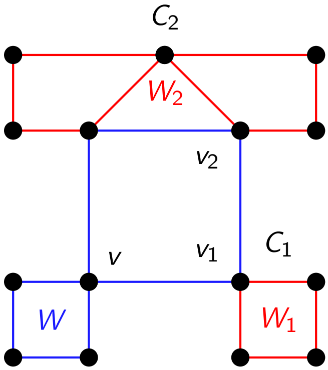

We have found vertices v_i which lie in both C_i and W.

Wlog, if G has r non-trivial components, W reaches first v_1, then v_2, and so on up to v_r. (Otherwise we can just reorder $C_1, \ldots, C_r$.)

So our idea works! We follow W until reaching v_1, then follow all of W_1 (returning to v_1), then follow W until reaching v_2, and so on. □

Note this gives us an algorithm!

Breather slide: Leonhard Euler (1707–1783)

- One of the greatest mathematicians of all time.
- Discovered foundational ideas in just about every single field of modern mathematics.
- Not only proved $e^{i\pi} + 1 = 0$, but introduced the notation for e, i and π.
- Over 800 papers and books written, constituting about **one third** of **all** research in maths **and** physics **and** engineering mechanics in 1725–1800.

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(i) $u = v$ and every vertex of G has even degree; or

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"Only if": ✓ **"If" case (i):** ✓

Suppose $u \neq v$, $d(u)$ and $d(v)$ are odd, and every other degree is even. We could use a very similar argument to part (i), but there's an easier way: we **reduce** the problem to part (i).

Form a new graph, G', by adding a new vertex w and edges from u to w and w to v. Then every vertex of G' has even degree, so G' contains an Euler walk W starting and ending at u by part (i).

Then we remove the subpath uwv from W, which turns it into an Euler walk from u to v in G. □

Again, this proof gives us an algorithm. So we know exactly which graphs have Euler walks, and we can find them quickly when they exist!