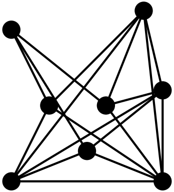
Shaking hands
COMS20010 (Algorithms II)

John Lapinskas, University of Bristol

The Handshake Lemma



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but adding up the vertex degrees would be much easier.

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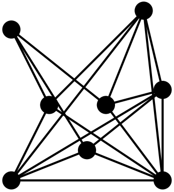

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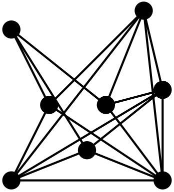

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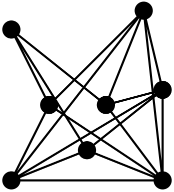

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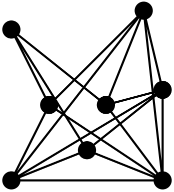

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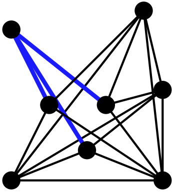

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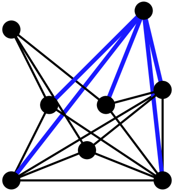

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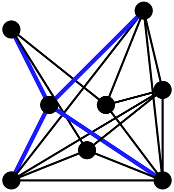

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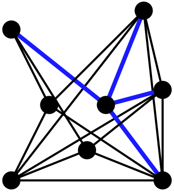

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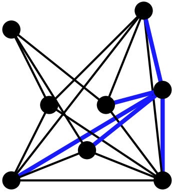

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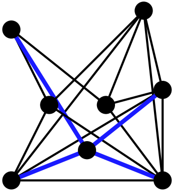

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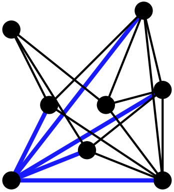

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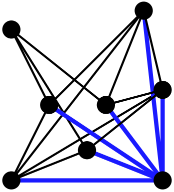

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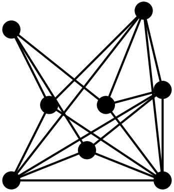

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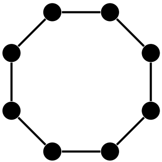
Here, $\displaystyle\sum_{v} d(v) = 3 + 5 + 4 + 4 + 5 + 4 + 5 + 6 = 36$, so 18 edges total.

Example applications

Handshake Lemma: For any graph $G = (V, E)$, $\sum_{v \in V} d(v) = 2|E|$.

Question: How many edges does an n-vertex cycle have?



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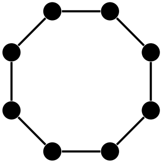

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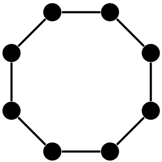

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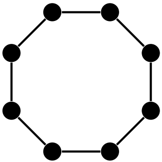

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Answer: No, as then $\sum_{v \in V} d(v)$ would be $3|V|$ (which is odd).
$2|E|$ is even, so this can't happen.

In **directed** graphs, can we express the number of edges in terms of in- and out-degrees? Yes!

Directed Handshake Lemma:

For any digraph $G = (V, E)$, $\sum_{v \in V} d^+(v) = \sum_{v \in V} d^-(v) = |E|$.

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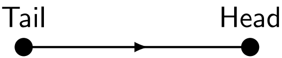

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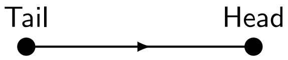

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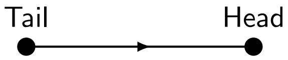

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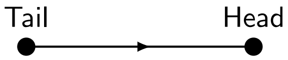

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Similarly, counting head-edge pairs gives $\sum_{v \in V} d^-(v) = |E|$. □