# Trees
# COMS20010 (Algorithms II)

John Lapinskas, University of Bristol

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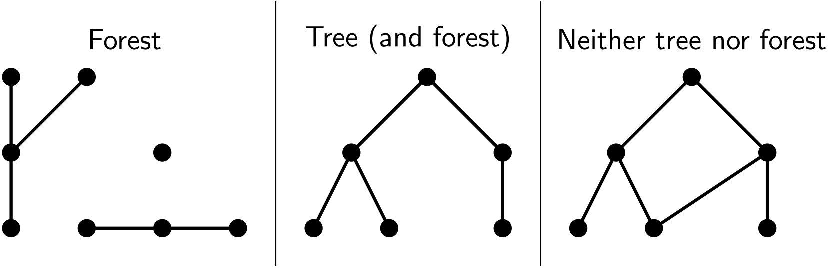

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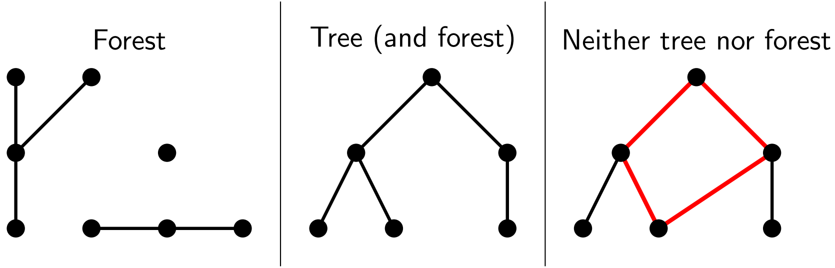

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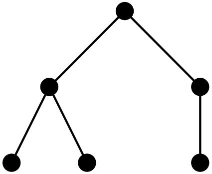

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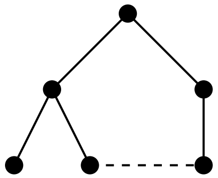

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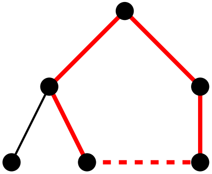

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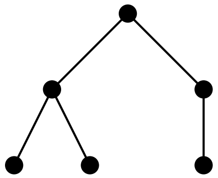

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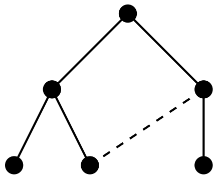

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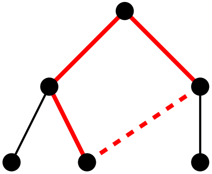

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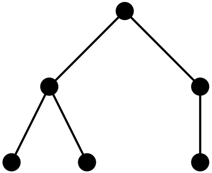

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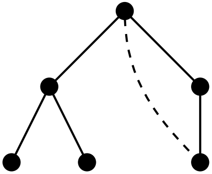

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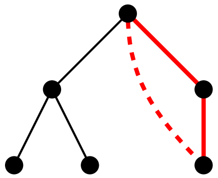

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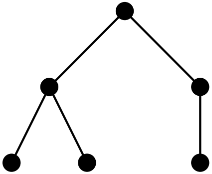

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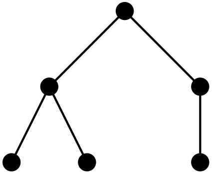

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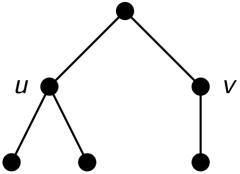


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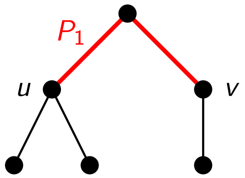


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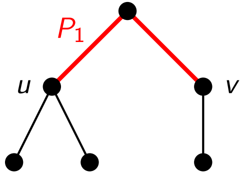


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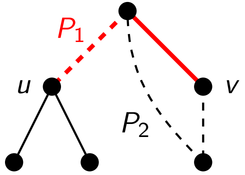


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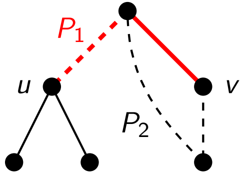


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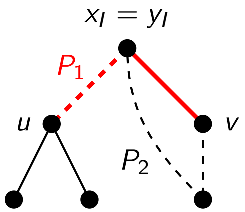


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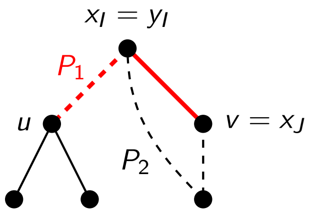


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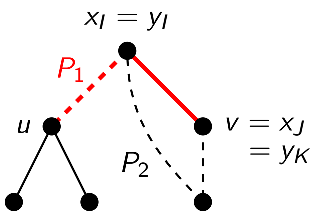


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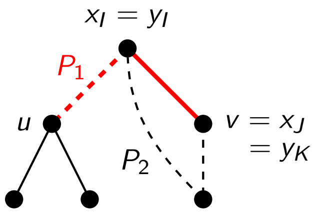


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Then $x_I x_{I+1} \dots x_J y_{K-1} y_{K-2} \dots y_I$ is a cycle, so $T$ is not a tree. □

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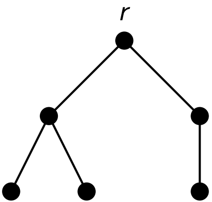

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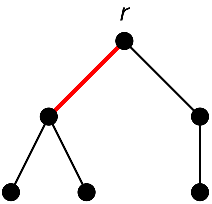

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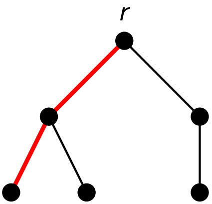

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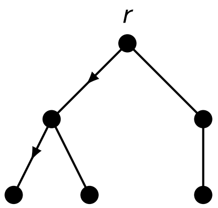

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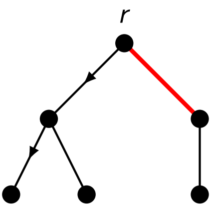

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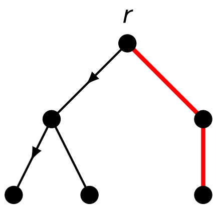

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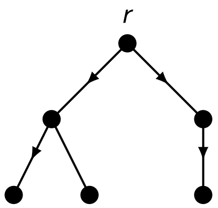

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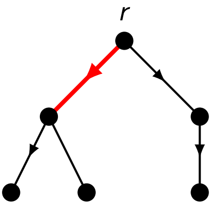

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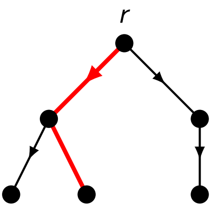

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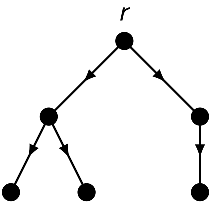

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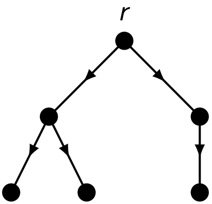

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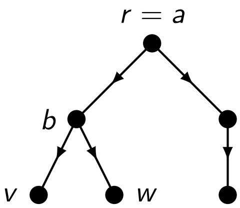


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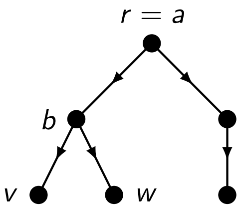


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And suppose $b$ is also on another path $P_w$.

Then both $P_v$ and $P_w$ must start with $P_b$, since $P_b$ is the **unique** path from $r$ to $b$.
So $P_w$ also directs $a \to b$. ✓

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**Lemma 1:** Any pair of vertices in a tree is joined by a **unique** path. □

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**Proof idea:** Take an arbitrary root $r \in V$. For all vertices $v$, let $P_v$ be the unique path from $r$ to $v$. Direct $T$'s edges along these paths. ✓

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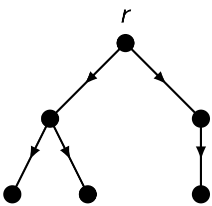

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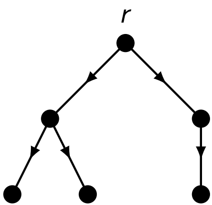


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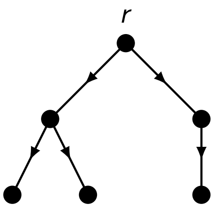


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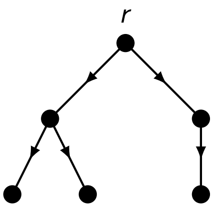


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**Bonus:** We also just defined rooted trees in terms of graphs.

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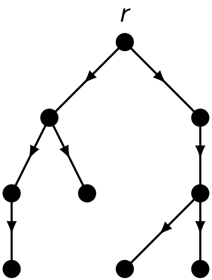


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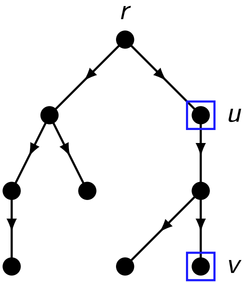


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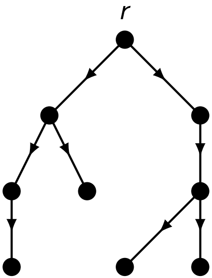


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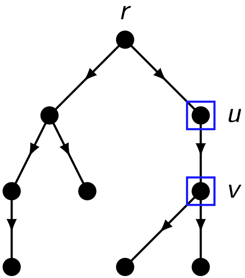


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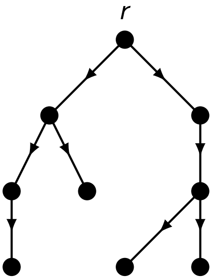


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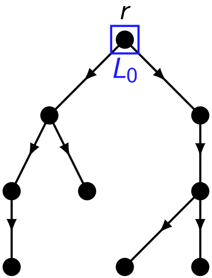


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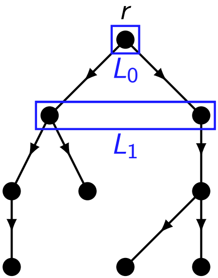


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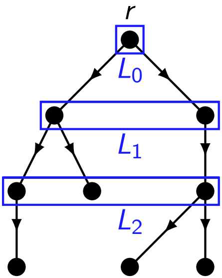


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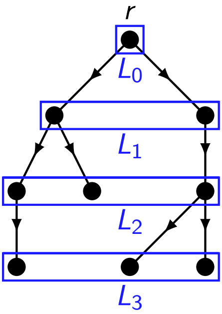


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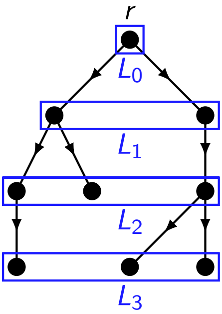


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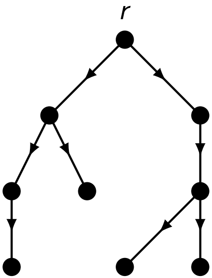


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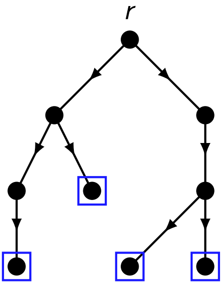


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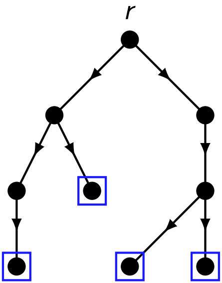


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Plugging this in gives $|E|=n-1=\frac{1}{2}\sum_{v\in V}d(v)\geq n-\frac{x}{2}$.

Solving for $x$ gives $x\geq 2$, so we're done! □

# The Fundamental Lemma of Trees

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When you're actually working with trees, it's good to have one single result that tells you that all the "obvious" things are true. This is that result.

**Lemma:** The following are equivalent for an $n$-vertex graph $T = (V, E)$:

(A) $T$ is connected and has no cycles, i.e. is a tree;

(B) $T$ has $n-1$ edges and is connected;

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We've already proved (A) $\Rightarrow$ (D) (Lemma 1)...

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**Lemma:** The following are equivalent for an $n$-vertex graph $T = (V, E)$:

(A) $T$ is connected and has no cycles, i.e. is a tree;

(B) $T$ has $n-1$ edges and is connected;

(C) $T$ has $n-1$ edges and has no cycles;

(D) $T$ has a unique path between any pair of vertices.

We've already proved (A) $\Rightarrow$ (D) (Lemma 1)...
as well as (A) $\Rightarrow$ (B) and (A) $\Rightarrow$ (C) (Lemma 2).

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**(A) $\Rightarrow$ (B), (C) and (D):** ✓

---

## (D) $\Rightarrow$ (A):

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**(D) $\Rightarrow$ (A):** $T$ has a path between any pair of vertices, so it's connected.

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**(A) $\Rightarrow$ (B), (C) and (D):** ✓

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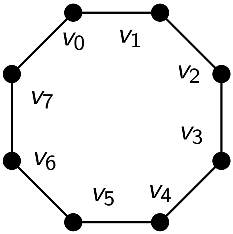
**(D) $\Rightarrow$ (A):** $T$ has a path between any pair of vertices, so it's connected.

And on any cycle $v_0 \dots v_k$, there are two different paths from $v_0$ to $v_k$:



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**(A) ⇒ (B), (C) and (D):** ✓

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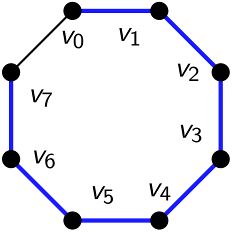


- the path $v_0 \dots v_k$; and

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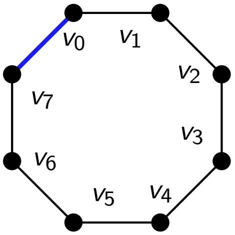


- the path $v_0 \dots v_k$; and
- the edge $v_0 v_k$.

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**(A) $\Rightarrow$ (B), (C) and (D):** ✓

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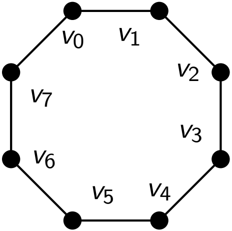


- the path $v_0 \dots v_k$; and
- the edge $v_0 v_k$.

So $T$ has no cycles. ✓

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**(A) ⇒ (B), (C) and (D):** ✓ **(D) ⇒ (A):** ✓

---

**(C) ⇒ (A):** Suppose $T$ has no cycles and components $C_1, \dots, C_r$.

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**(A) ⇒ (B), (C) and (D):** ✓ **(D) ⇒ (A):** ✓

---

**(C) ⇒ (A):** Suppose $T$ has no cycles and components $C_1, \ldots, C_r$.

Each of these components has no cycles, and is connected, so it's a tree. So by (A) ⇒ (B) (or Lemma 2), each $C_i$ has $|V(C_i)| - 1$ edges.

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**(C) ⇒ (A):** Suppose $T$ has no cycles and components $C_1, \dots, C_r$.

Each of these components has no cycles, and is connected, so it's a tree. So by (A) ⇒ (B) (or Lemma 2), each $C_i$ has $|V(C_i)| - 1$ edges.

Every edge of $T$ is in some $C_i$, so $|E| = \sum_i (|V(C_i)| - 1) = n - r$. But we know $|E| = n - 1$, so we must have $r = 1$. ✓

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**(B) $\Rightarrow$ (A):** We will need to use:

**Claim:** If $T = (V, E)$ is connected, and $e \in E$ is on a cycle, then $T - e$ is connected.

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**Claim:** If $T = (V, E)$ is connected, and $e \in E$ is on a cycle, then $T - e$ is connected.

**Proof from Claim:** Suppose $T$ is not a tree, so it has a cycle.

We form a new graph $T'$ by repeatedly removing edges from cycles in $T$ (in arbitrary order) until no more cycles remain.

Then $T'$ has no cycles, and the Claim implies it's connected, so it's a tree. So by (A) ⇒ (B) (or Lemma 2), $T'$ has $n - 1$ edges.

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So $T$ must have had **more than** $n - 1$ edges — a contradiction. □

**Claim:** If $T = (V, E)$ is connected, and $e \in E$ is on a cycle, then $T - e$ is connected.

**Claim:** If $T = (V, E)$ is connected, and $e \in E$ is on a cycle, then $T - e$ is connected.

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For all $a, b \in V$, we must find a path from $a$ to $b$ in $T - e$.

Let $P = x_0 \ldots x_k$ be a path from $a$ to $b$ in $T$.

**Claim:** If $T = (V, E)$ is connected, and $e \in E$ is on a cycle, then $T - e$ is connected.

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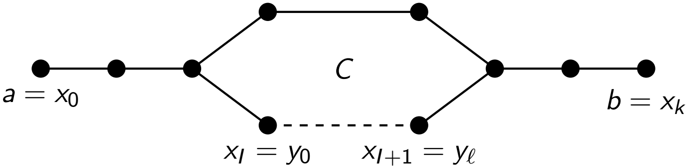
**If $e$ is in $P$:** Write $e = \{x_I, x_{I+1}\}$. Let $C = y_0 \dots y_\ell$ be a cycle in $T$ containing $e$ — without loss of generality we can take $y_0 = x_I$ and $y_\ell = x_{I+1}$.



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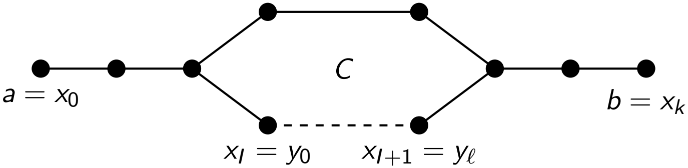


Then $x_0 \dots x_I y_1 \dots y_\ell x_{I+2} \dots x_k$ is a walk from $a$ to $b$ in $T - e$. Any walk from $a$ to $b$ contains a path from $a$ to $b$ (see quiz 2), so we're done. ✓

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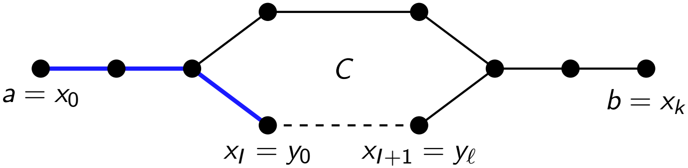


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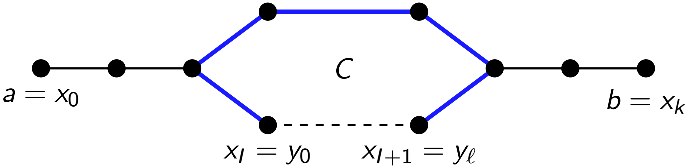


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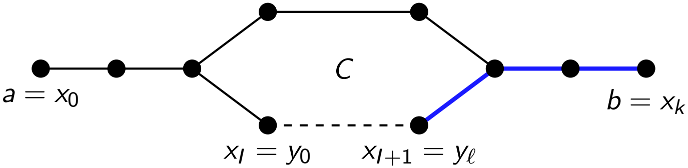


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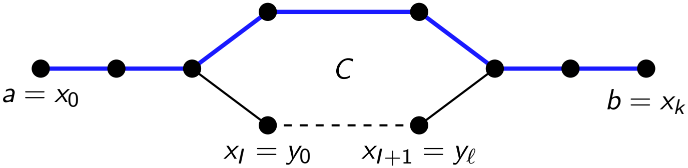


Then $x_0 \dots x_I y_1 \dots y_\ell x_{I+2} \dots x_k$ is a walk from $a$ to $b$ in $T - e$. Any walk from $a$ to $b$ contains a **path** from $a$ to $b$ (see quiz 2), so we're done. ✓

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Our reward for proving this lemma is:

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Our reward for proving this lemma is: we never have to think about basic tree properties in this level of detail ever again. (Except on the exam!)

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And there was much rejoicing.