Graph representations
COMS20010 (Algorithms II)

John Lapinskas, University of Bristol

Adjacency matrices

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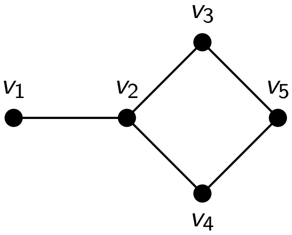

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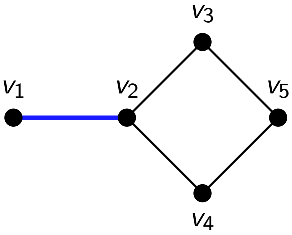

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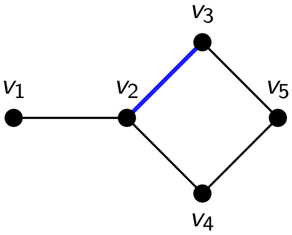

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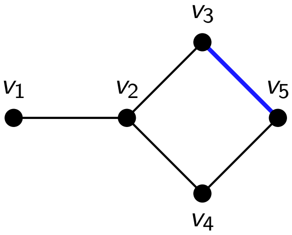

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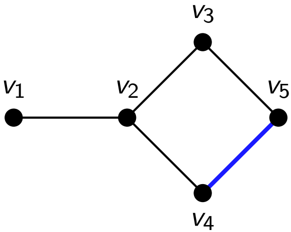

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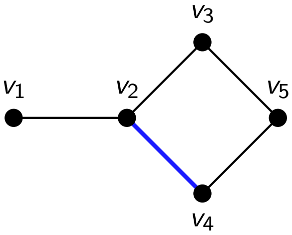

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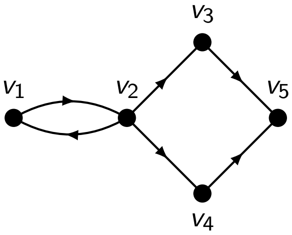

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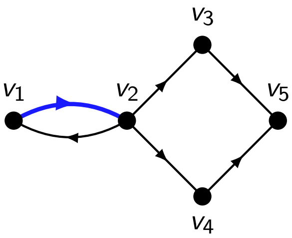

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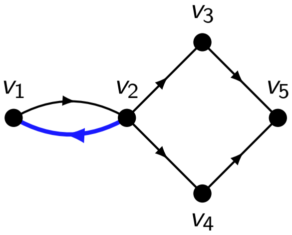

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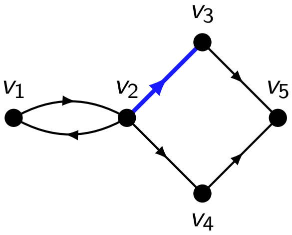

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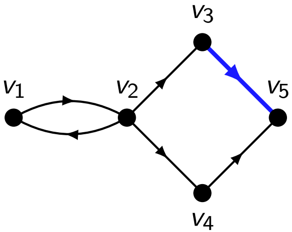

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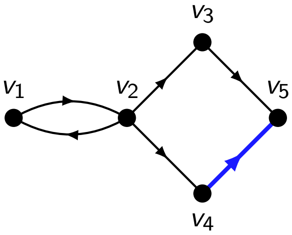

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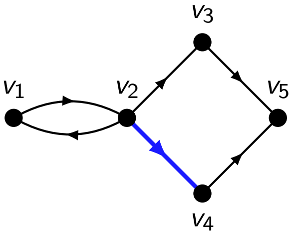

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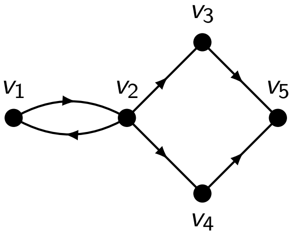

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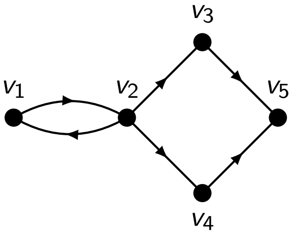

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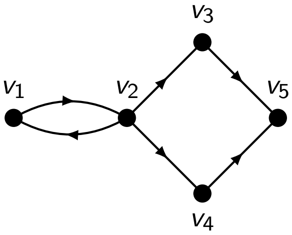

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A **neighbourhood query** ("What is $N^+(u)$?") takes $\mathbf{\Theta(|V|)}$ **time**.

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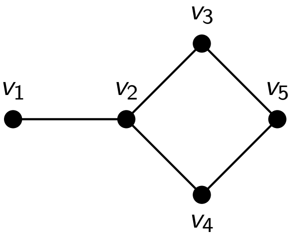
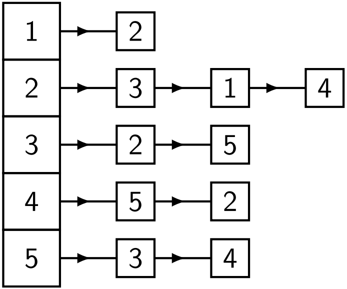
Another way is **adjacency list format**.

Here we store an array of $|V|$ linked lists $L_1, \dots, L_n$, where L_i contains the list of vertices v_j with $(v_i, v_j) \in E$ (in some order):





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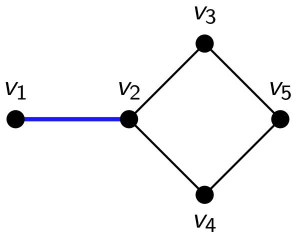

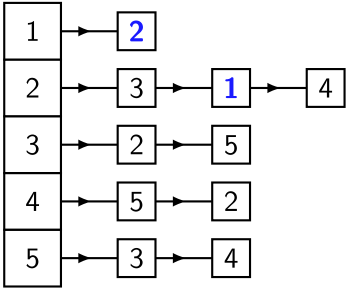

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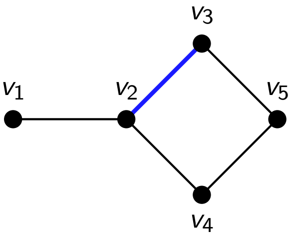

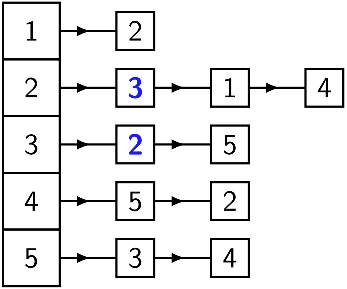

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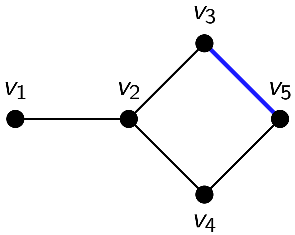

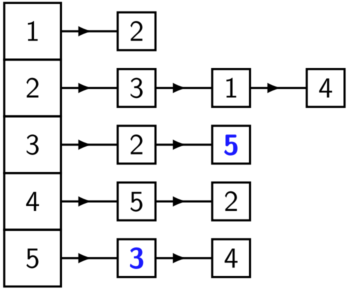

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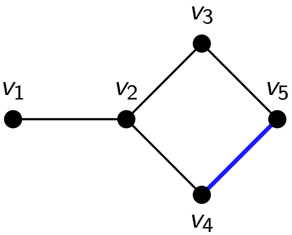

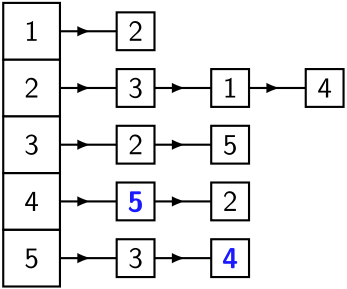

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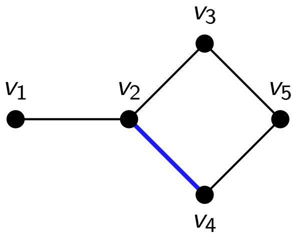

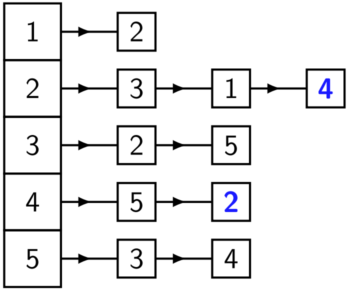

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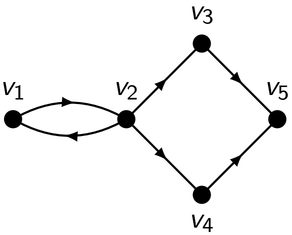

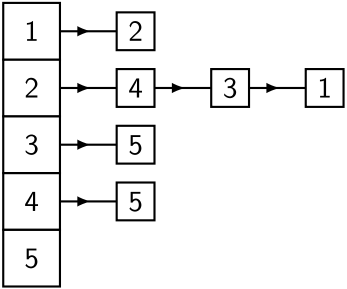

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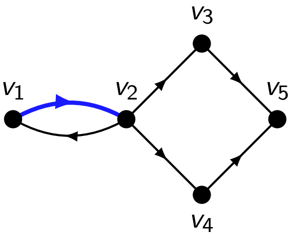

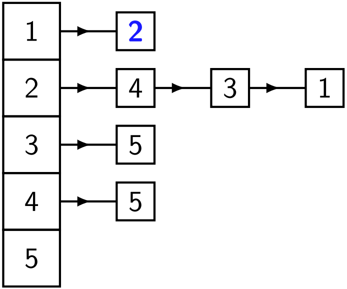

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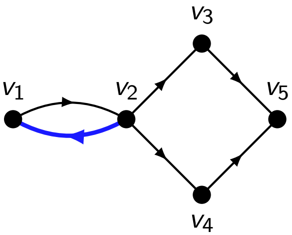

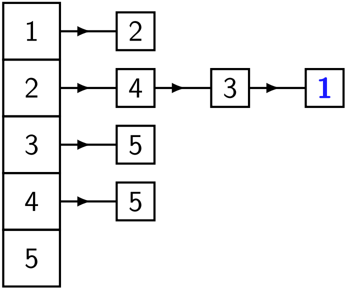

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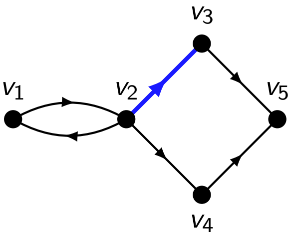

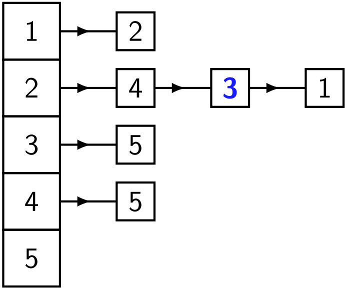

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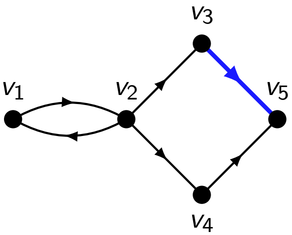

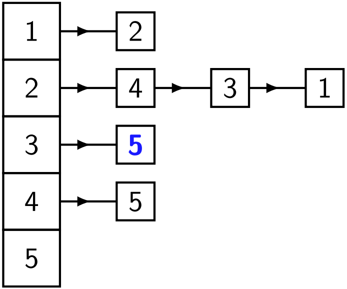

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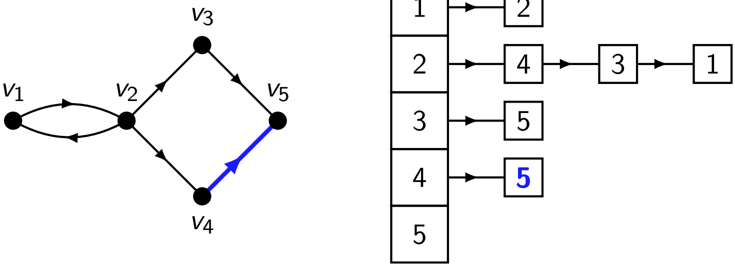

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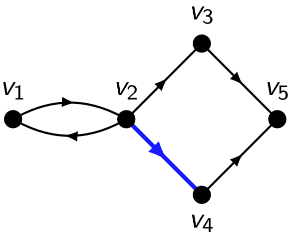

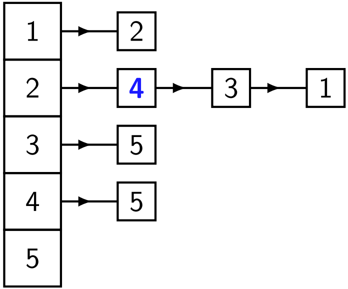

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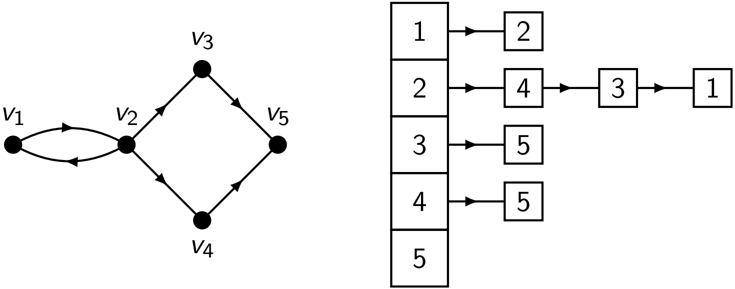

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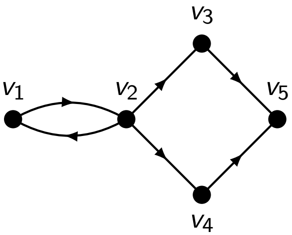

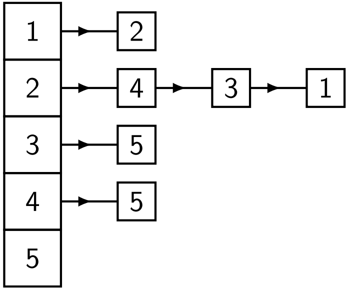

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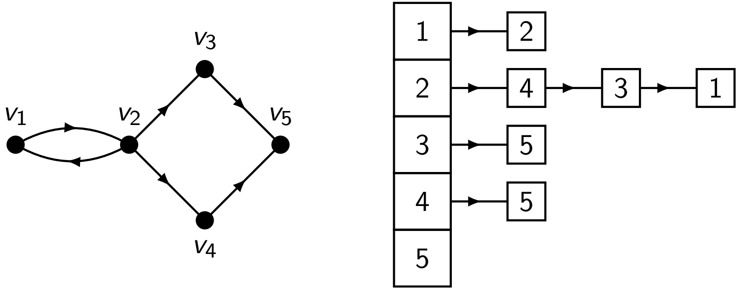

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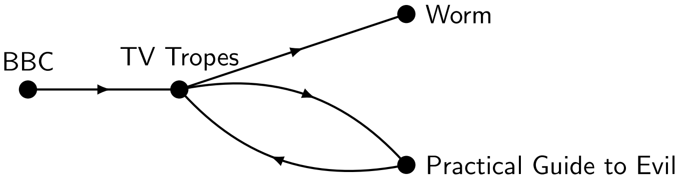

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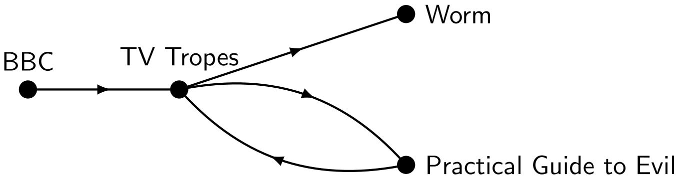

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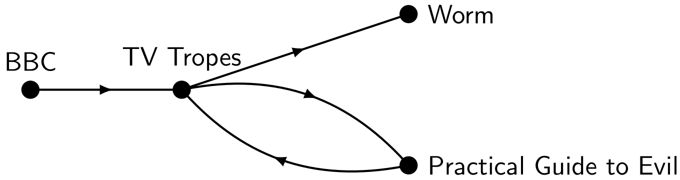

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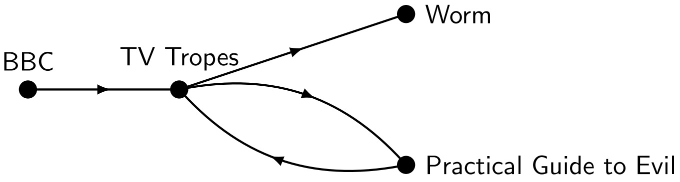

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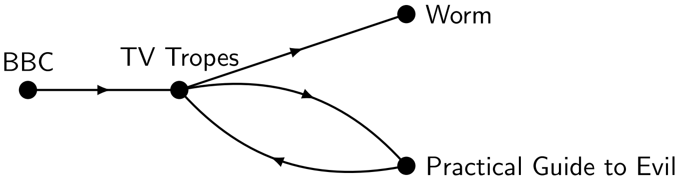

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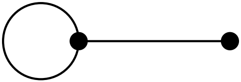
Situations like this, where the graph is only stored implicitly, are why we really care about the adjacency list and matrix models.

Loops and multiple edges

Sometimes it's useful to consider graphs with:



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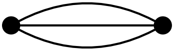

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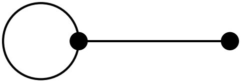

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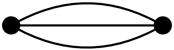

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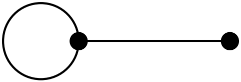

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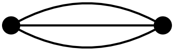

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So in this course we will only consider standard (a.k.a. **simple**) graphs, without loops or multiple edges.