Matchings I: Definitions
COMS20010 (Algorithms II)

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Definition by example

Suppose I'm trying to schedule a sequence of meetings with my tutees.
I can only see one of them at once, and each of them can make a different set of times:

Tutee	**Available times**
Aisha	2:00, 3:00, 3:30
Bob	2:30
Cara	2:00, 2:30, 4:00
Dale	2:00, 3:30, 4:00
Erica	2:00, 2:30, 3:00, 3:30, 4:00

Who should I see when?

We can represent this with a graph:

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Aisha	2:00, 3:00, 3:30
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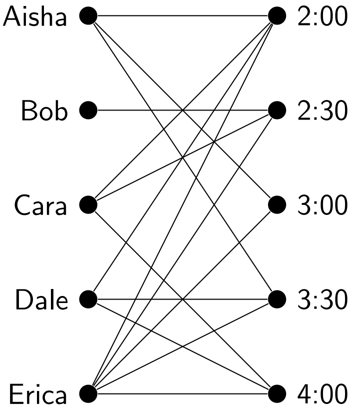

We join a student to a time if they can make that time.

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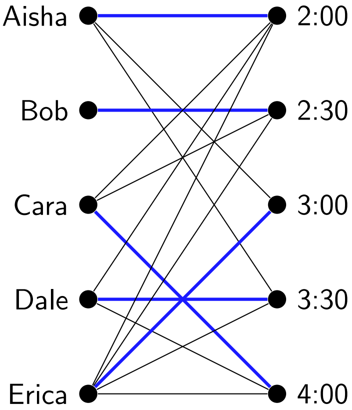
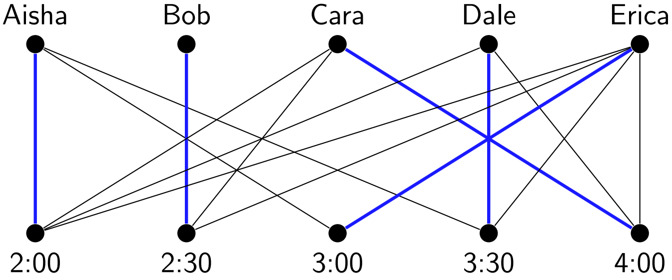


We join a student to a time if they can make that time.
A valid choice of times corresponds to a **matching** in the graph.

Formal definitions



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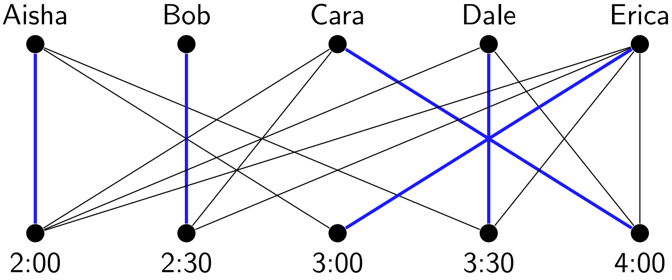

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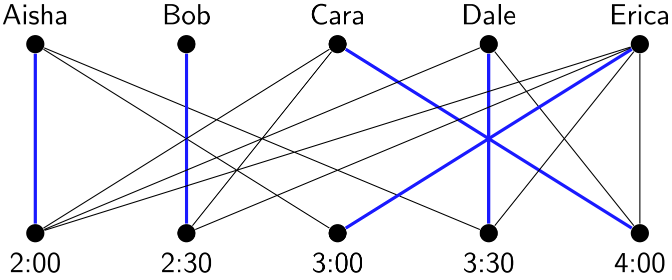

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$$A = \{\text{Aisha}, \text{Bob}, \text{Cara}, \text{Dale}, \text{Erica}\},\ B = \{2{:}00, 2{:}30, 3{:}00, 3{:}30, 4{:}00\}.$$

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Note that bipartitions are not unique! E.g. we could also take:

$$A = \{2{:}00, 2{:}30, 3{:}00, 3{:}30, 4{:}00\},\ \ B = \{\text{Aisha}, \text{Bob}, \text{Cara}, \text{Dale}, \text{Erica}\}.$$

(**Exercise:** Are there ever more than two possibilities?)

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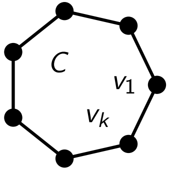

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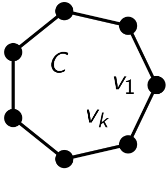

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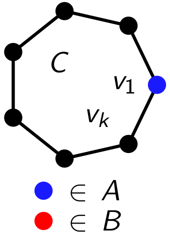

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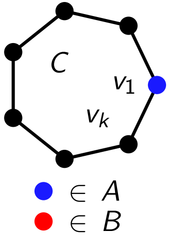

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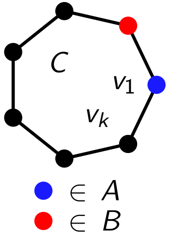

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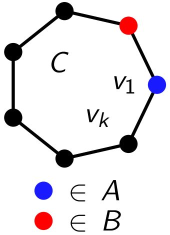

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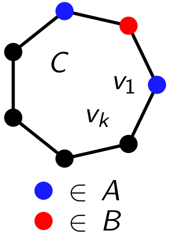

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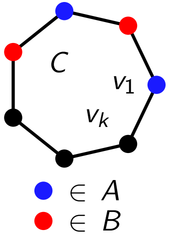

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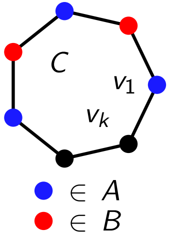

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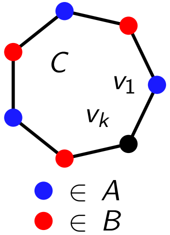

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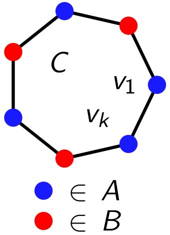

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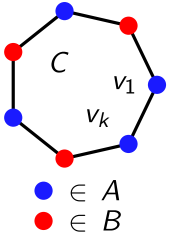
But v_1 and v_k are adjacent, so this is a contradiction. ✓



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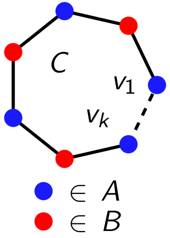

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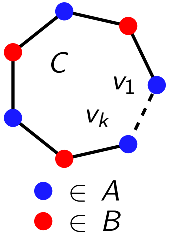

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"If": See problem sheet.

General problem statement: Given a bipartite graph $G = (V, E)$, output a matching which is as large as possible.

Algorithms for this problem (or generalisations) can be applied to e.g.:

- Scheduling meetings.
- Matching lectures to available rooms.
- Matching people to tasks they're qualified for.
- Matching residents to hospitals. (Must output a "stable" matching.)
- Matching banner ads to viewers. (Must be an "online" algorithm.)
- Matching people as a dating site. (Must handle non-bipartite graphs!)