Matchings II: Finding the maximum
COMS20010 (Algorithms II)

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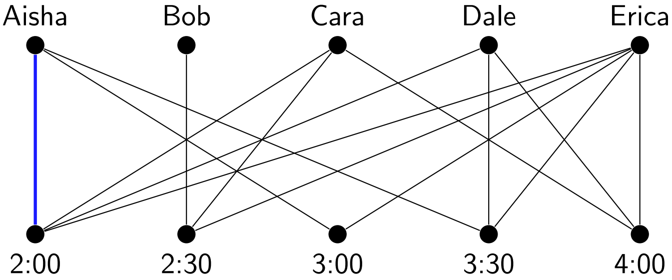
An algorithm for maximum matchings

General problem statement: Given a bipartite graph $G = (V, E)$, output a matching M which is as large as possible (i.e. **maximum**).

Can we form M by greedily adding edges? E.g.:



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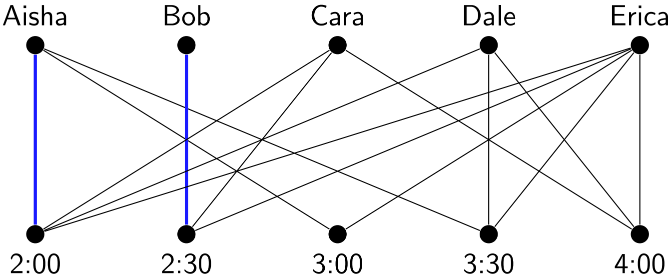

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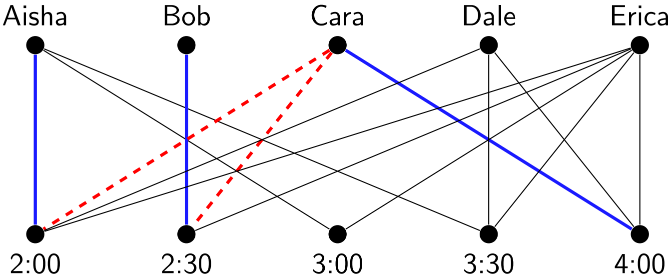

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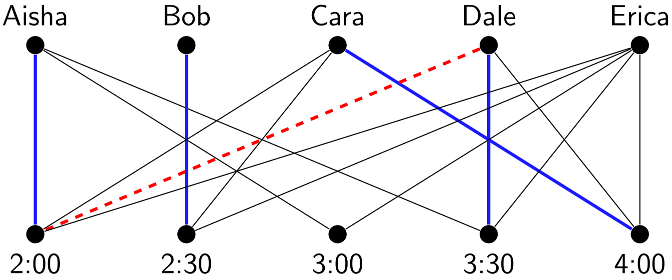

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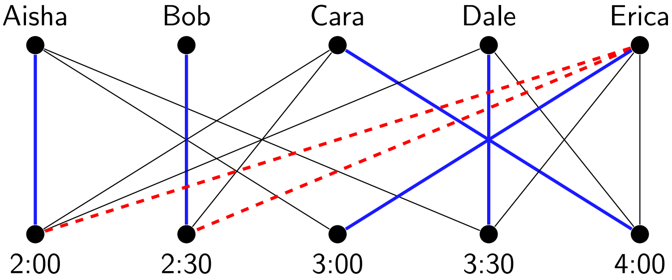

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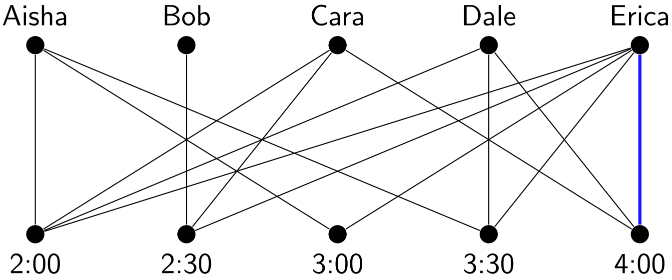

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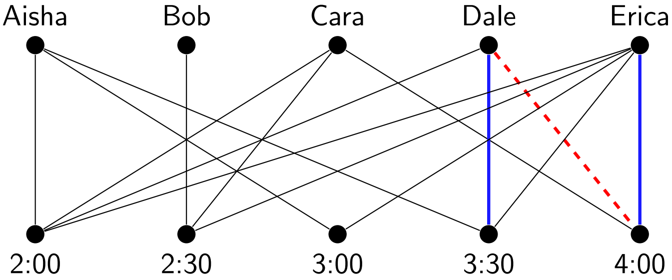

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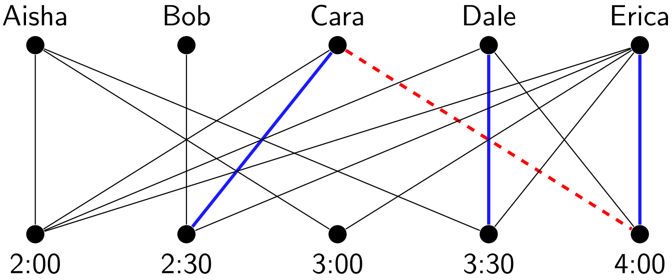

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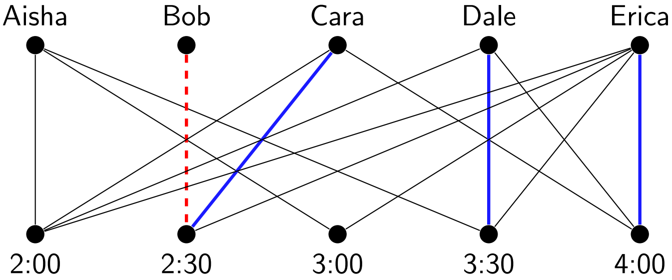
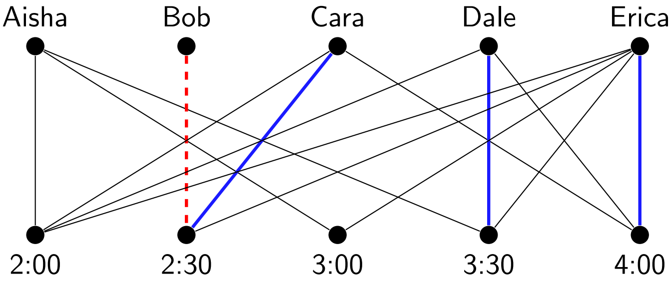


But if we had considered edges a different order... we wouldn't have been able to match Bob! So this algorithm **fails**.

Repairing poor decisions: an example

But maybe all is not lost... Say we try to force Bob into the matching.



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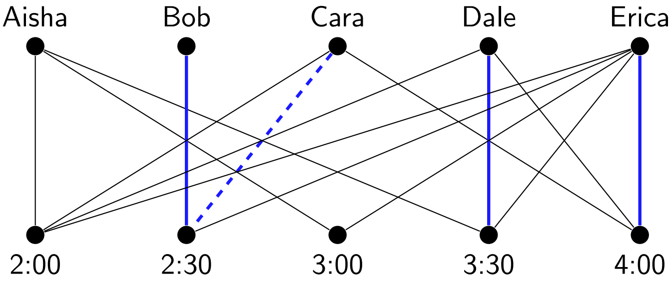

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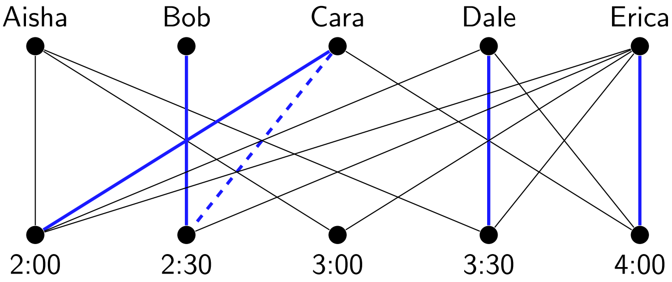

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Which leaves us free to rematch Cara...

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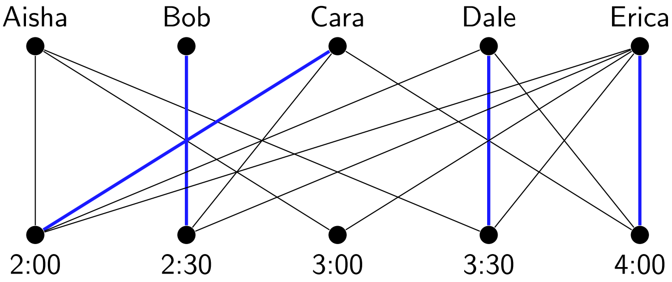

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Who can still meet us at 2:00. So we succeeded in matching Bob!

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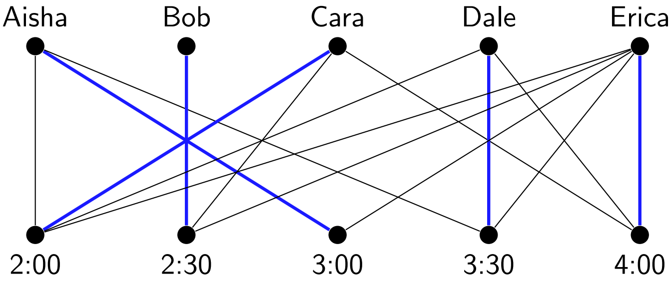
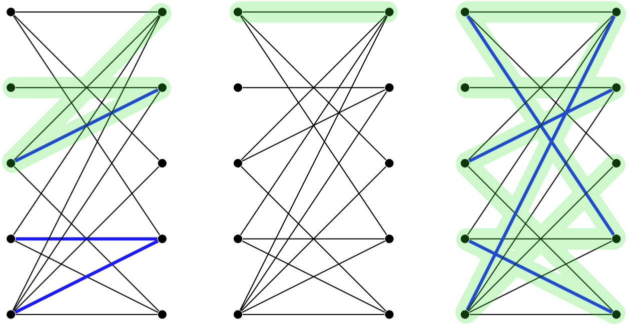


This forces us to unmatch 2:30...
Which leaves us free to rematch Cara...
Who can still meet us at 2:00. So we succeeded in matching Bob!
And now we continue as before, and get a perfect matching.

Repairing poor decisions: the general method

Given a matching M in a bipartite graph G, an **augmenting path** P for M is a path in G which alternates between matching and non-matching edges, and which begins and ends with unmatched vertices.

Formally, writing $P = v_0 \dots v_k$, we require $\{v_i, v_{i+1}\} \in M$ for all odd i, $\{v_i, v_{i+1}\} \notin M$ for all even i, and $v_0, v_k \notin \bigcup_{e \in M} e$. For example:



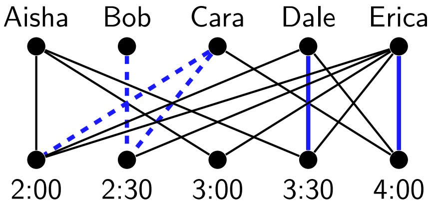
Given a matching M in a bipartite graph G, an **augmenting path** for M is a path $P = v_0 \dots v_k$ such that:

- $\{v_i, v_{i+1}\} \in M$ for all odd i;
- $\{v_i, v_{i+1}\} \notin M$ for all even i;
- $v_0, v_k \notin \bigcup_{e \in M} e$.

If P is an augmenting path for M, we define

$$\mathrm{Switch}(M, P) = M - \{\{v_i, v_{i+1}\} : i \text{ is odd}\} \cup \{\{v_i, v_{i+1}\} : i \text{ is even}\}.$$

Then $\mathrm{Switch}(M, P)$ is a matching containing one more edge than M.



This suggests a new greedy algorithm!

A **correct** algorithm for maximum matchings

Algorithm: MaxMatching (Sketch)

Input : A bipartite graph $G = (V, E)$.
Output : A list of edges forming a matching in G of maximum size.

begin
  Initialise $M \leftarrow []$, the empty matching.
  while *G contains an augmenting path for M* **do**
    Find an augmenting path P for M.
    Update $M \leftarrow \mathrm{Switch}(M, P)$.
  Return M.

To make this work, we need to do two things:

- Find an efficient way to find an augmenting path whenever one exists.
- Prove that if M has no augmenting paths, then M is maximum.

Finding augmenting paths efficiently

If we search by brute force, this could take $\Theta(|V|!)$ time! Let's not.

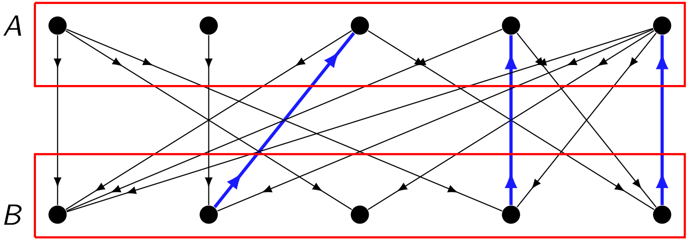
One general theme of this course: solve a complex problem by applying an algorithm for a simple problem in a clever way. We call this **reducing** the complex problem to the simple one.

Here, we reduce the problem of finding an augmenting path to a problem we can already solve: finding a path from one set to another in a **directed** graph, via breadth-first search.

(For how to apply breadth-first search to sets instead of vertices, see last week's problem sheet — this is itself a reduction!)

Suppose $G = (V, E)$ has a matching M and a bipartition (A, B). Turn G into an auxiliary digraph $D_{G,M}$ by directing non-matching edges from A to B and matching edges from B to A. Formally:

$$\begin{aligned} V(D_{G,M}) &:= V, \\ E(D_{G,M}) &:= \{(a, b)\colon a \in A,\ b \in B,\ \{a, b\} \in E \setminus M\} \cup \\ &\qquad\qquad \{(b, a)\colon a \in A,\ b \in B,\ \{a, b\} \in M\}. \end{aligned}$$



$D_{G,M}$ is defined by directing edges outside M from A to B,
and edges in M from B to A.

$P = v_0 \dots v_k$ is **augmenting** if $\{v_i, v_{i+1}\} \in M$ for all odd i,
$\{v_i, v_{i+1}\} \notin M$ for all even i, and $v_0, v_k \notin \bigcup_{e \in M} e$.

Let $U = V \setminus \bigcup_{e \in M} e$ be the set of vertices not matched by M.

Lemma: A path in G is augmenting for M if and only if it's also a path from $U \cap A$ to $U \cap B$ in $D_{G,M}$.

Proof: First note any augmenting path in G has endpoints in $U \cap A$ and $U \cap B$, since it has an odd number of edges and G is bipartite.

So let $P = v_0 \dots v_k$ be any path in G with $v_0 \in U \cap A$, $v_k \in U \cap B$. We show P is augmenting for M iff it is also a path in $D_{G,M}$.

- G is bipartite $\Rightarrow v_i \in A$ for all even i and $v_i \in B$ for all odd i. So:
- $\{v_i, v_{i+1}\} \in M$ for all odd $i \Leftrightarrow (v_i, v_{i+1}) \in E(D_{G,M})$ for all odd i;
- $\{v_i, v_{i+1}\} \notin M$ for all even $i \Leftrightarrow (v_i, v_{i+1}) \in E(D_{G,M})$ for all even i. $\square$

Algorithm: MaxMatching

Input : A bipartite graph $G = (V, E)$.

Output : A list of edges forming a matching in G of maximum size.

begin

  Find a bipartition (A, B) of G. Initialise $M \leftarrow []$.

  repeat

    Form the graph $D_{G,M}$.

    Set P to be a path from $U \cap A$ to $U \cap B$ in $D_{G,M}$ if one exists. Otherwise, **break**.

    Update $M \leftarrow \text{Switch}(M, P)$.

  Return M.

Invariant: At the start of the ith loop iteration, M is a matching with $i - 1$ edges. M can have at most $|V|/2$ edges in total, so MaxMatching outputs a matching with no augmenting paths.

Algorithm: MaxMatching

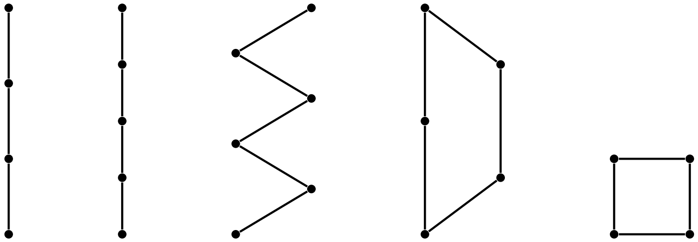
1 **begin**
2   Find a bipartition (A, B) of G. Initialise $M \leftarrow []$.
3   **repeat**
4   Form the graph $D_{G,M}$.
5   Set P to be a path from $U \cap A$ to $U \cap B$ in $D_{G,M}$ if one exists. Otherwise, **break**.
6   Update $M \leftarrow \text{Switch}(M, P)$.
7   Return M.

- Steps 2, 4 and 6 can all be done in $O(|E|)$ time. (Exercise!)
- Step 5 can be done in $O(|E|)$ time using breadth-first search, if G is in adjacency-list form.
- Steps 4–6 repeat at most $|V|$ times.

So overall the running time is $O(|E||V|)$.

Berge's Lemma: M has no augmenting paths $\Rightarrow$ M is maximum.
We suppose M is **not** maximum, and find an augmenting path.

Let M' be another matching which **is** maximum, so $|M'| > |M|$.
Consider the symmetric difference $S = M \triangle M'$, i.e. the graph formed of edges contained in either M or M' but not both.



Since each vertex is in at most one M edge and at most one M' edge, S has maximum degree at most 2.

So S is a disjoint union of path and cycle components.

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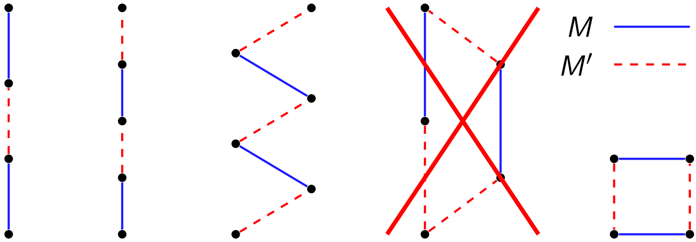

Since M and M' are matchings, each component's edges must alternate between M' and M. (In particular, no odd cycles!)

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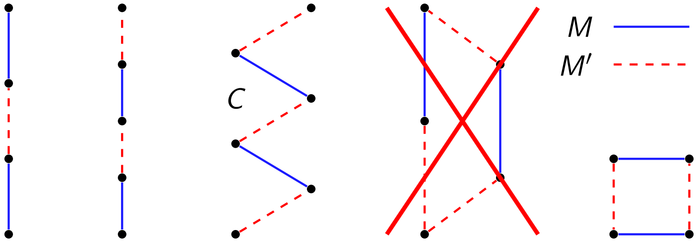

Since $|M'| > |M|$, some component C has more M'-edges than M-edges.
Since M'-edges and M-edges alternate, it has exactly **one** more M'-edge.

G is bipartite, so it has no odd cycles, so C must be a path starting and ending with an M'-edge — an augmenting path. $\square$

Recall that given a graph G, we proved that $\textsc{MaxMatching}$ returns a matching M for G with no augmenting path in time $O(|E||V|)$.

Berge's Lemma tells us that M is maximum, so we're done!

Let us celebrate with a matching pair of kittens.

D'awww.