2-3-4 trees I: Search and insertion
COMS20010 (Algorithms II)

John Lapinskas, University of Bristol

The limitations of binary search

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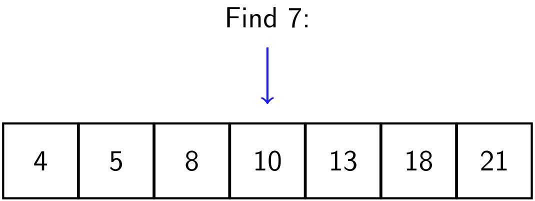

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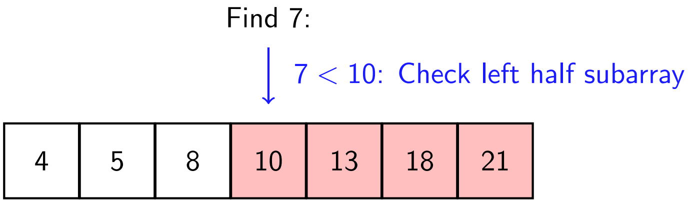

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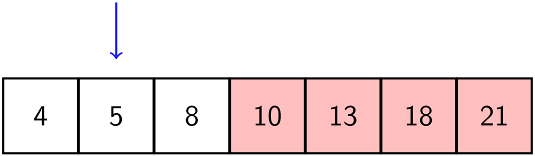

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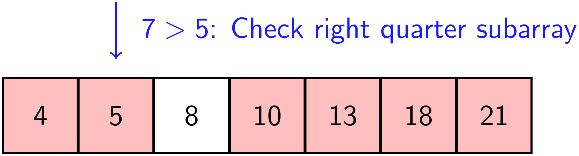

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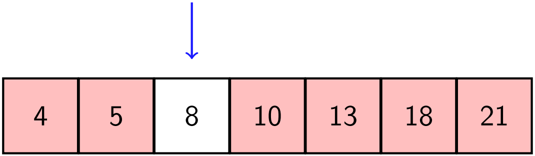

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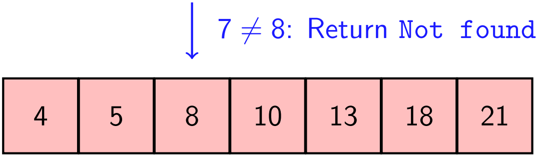

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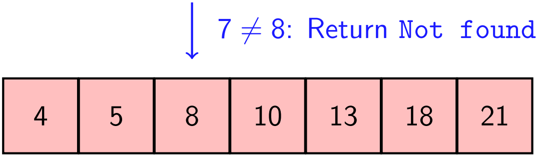

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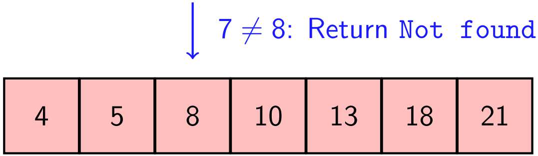

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Find 7:

$7 \neq 8$: Return `Not found`

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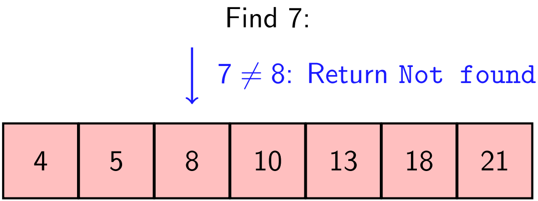

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Because we can't easily insert or remove things from the middle of the array — this takes $\Omega(n)$ time! And if we used a linked list instead... it would take $\Omega(n)$ time to find the halfway point.

Instead, we can use a **binary search tree**.

Binary search trees

Idea: Each node has 0–2 children. If a node's value is x, then **all** its left descendants' values are $< x$, and **all** its right descendants' values are $> x$.

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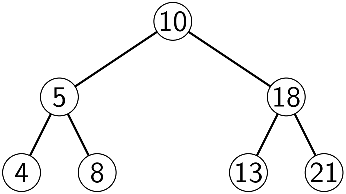

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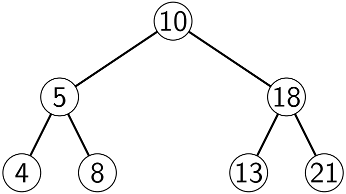

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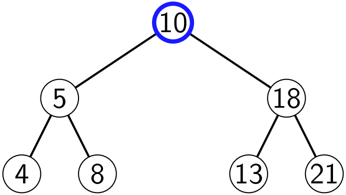

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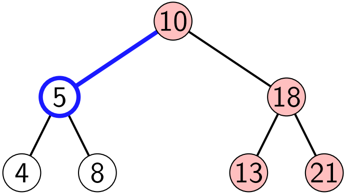

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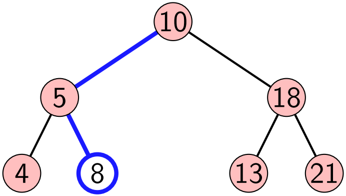

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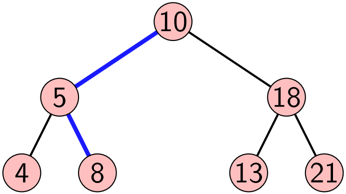

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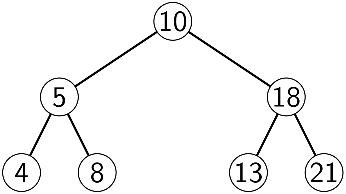

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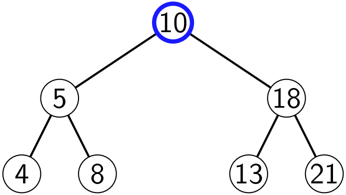

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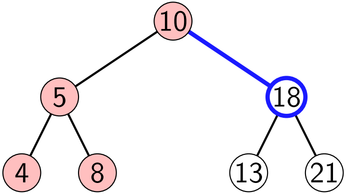

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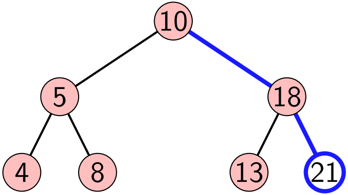

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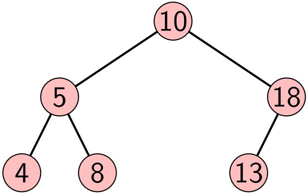

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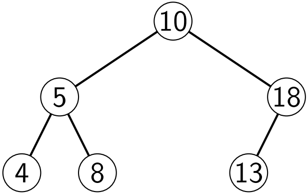

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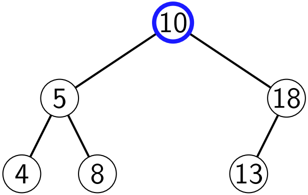

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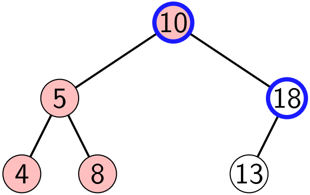

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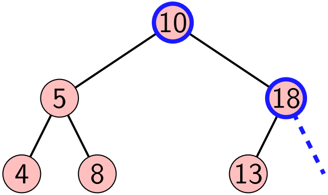

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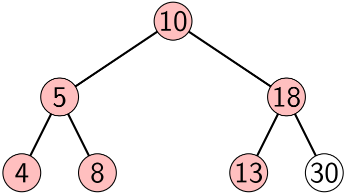

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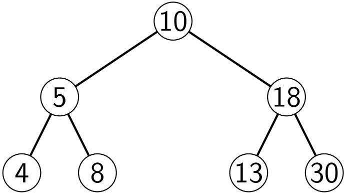

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$$n \approx 2^d + 2^{d-1} + \cdots + 1 = 2^{d+1} - 1 \Rightarrow d \in \Theta(\log n).$$

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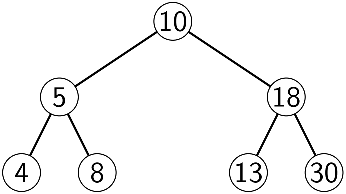

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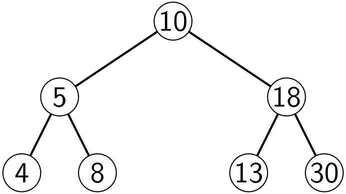

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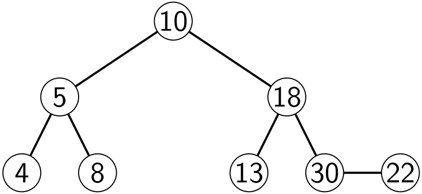

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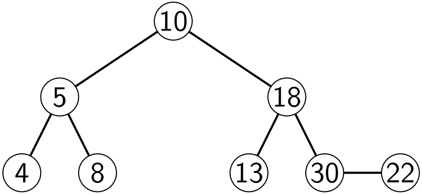

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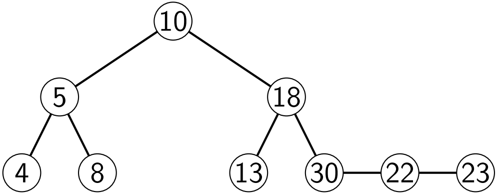

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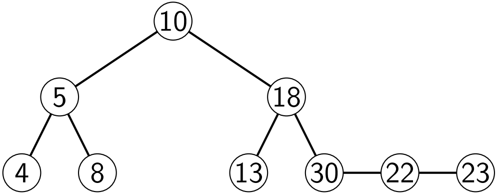

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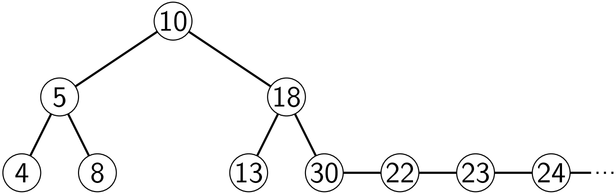

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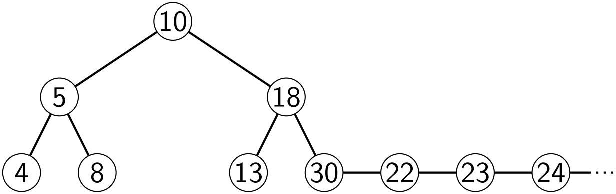

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Problem: What if that doesn't happen? We could get $d \in \Omega(n)$...

2-3-4 trees

Idea: Force the tree to be **perfectly** balanced, with all levels full. To make this possible to maintain, allow nodes to contain more than one value.

A ***k*-node** can have up to k children and contain $k-1$ values (so a binary search tree is made entirely of 2-nodes). We will allow $k \in \{2, 3, 4\}$.

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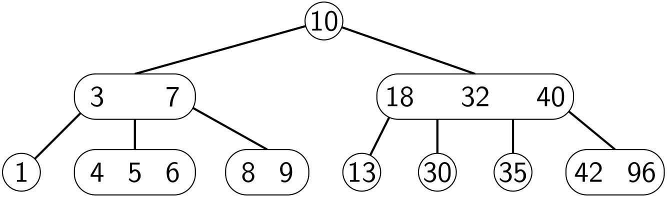

Say a 3-node has values $x_1 \le x_2$, and children c_1, c_2 and c_3.

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2-3-4 trees

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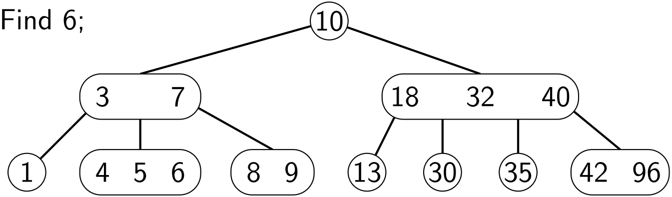
Find 6;



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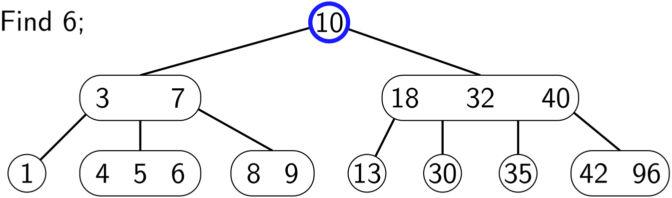

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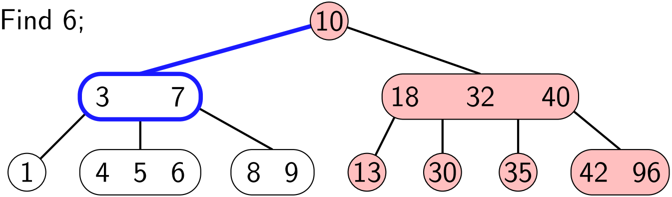

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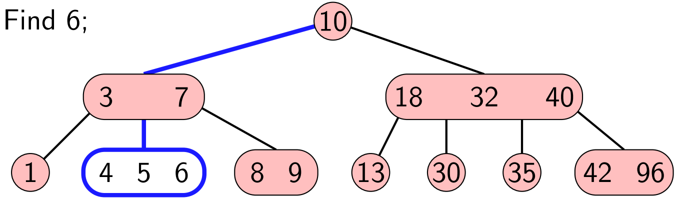

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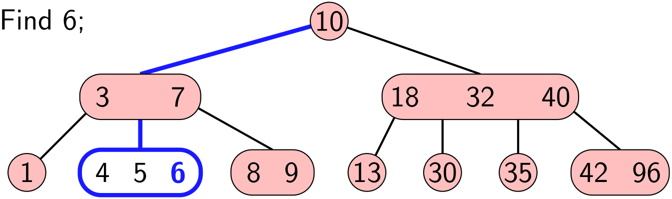

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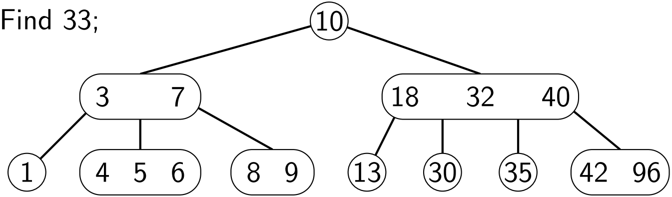

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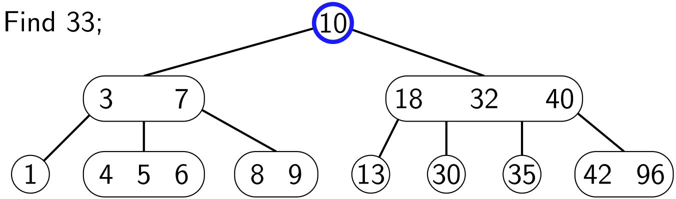

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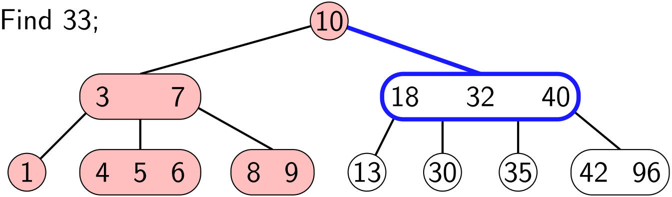

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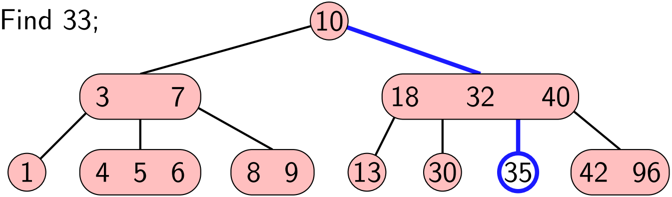

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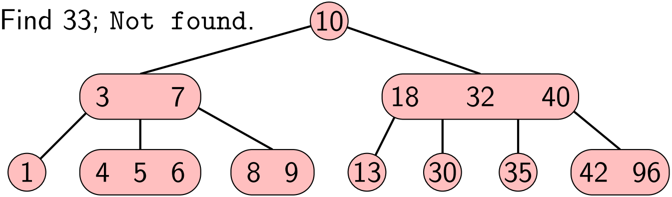
Find 33; `Not found`.



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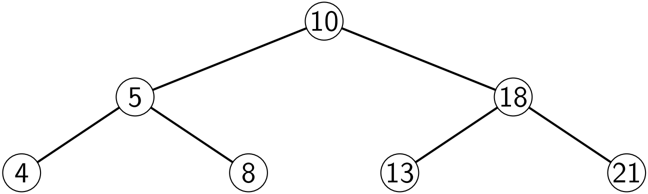
Inserting new values



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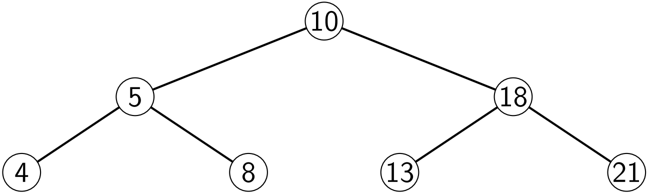
Inserting new values

Insert 22;



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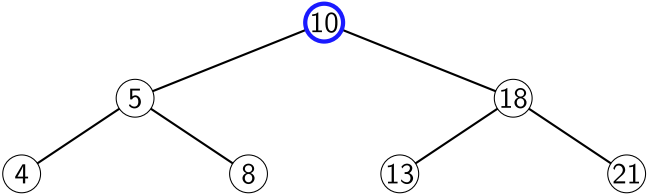

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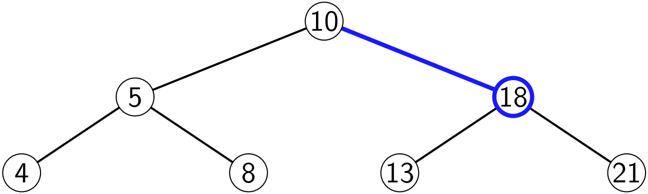

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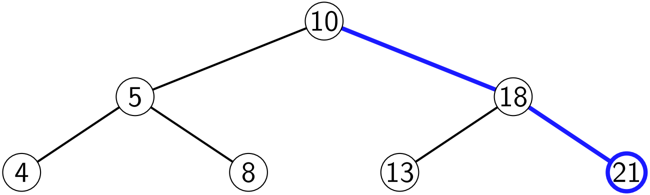

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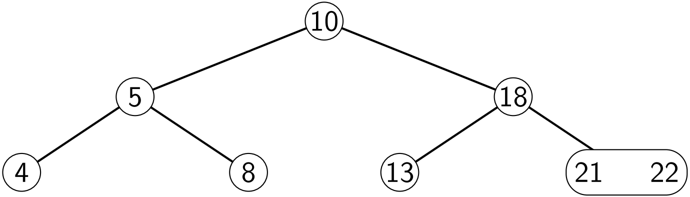

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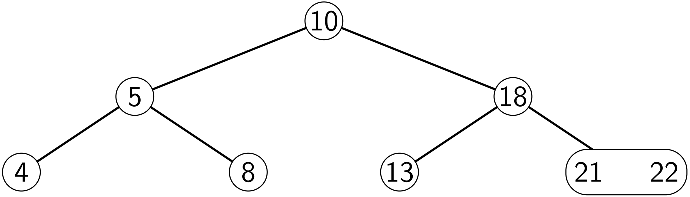

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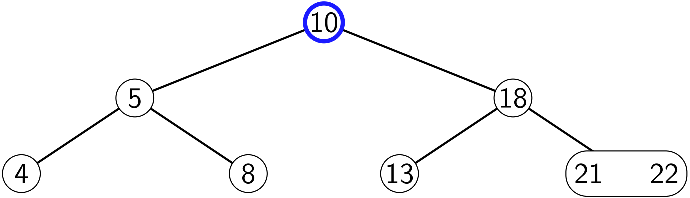

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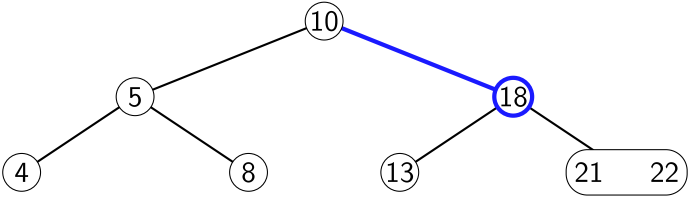

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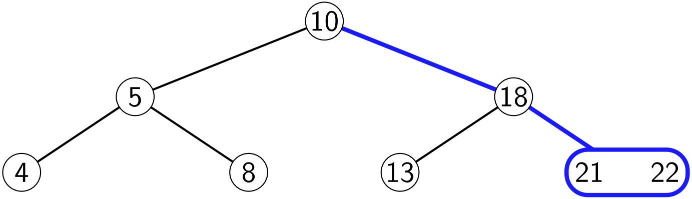

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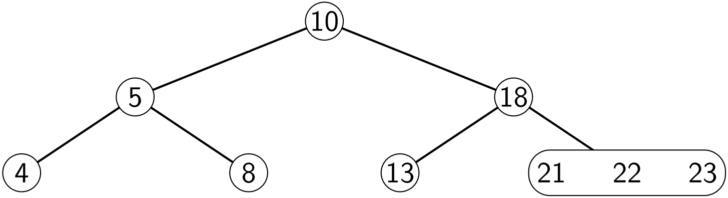

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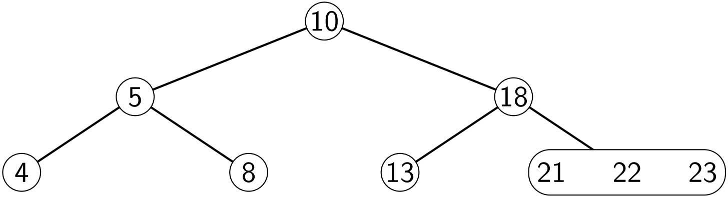

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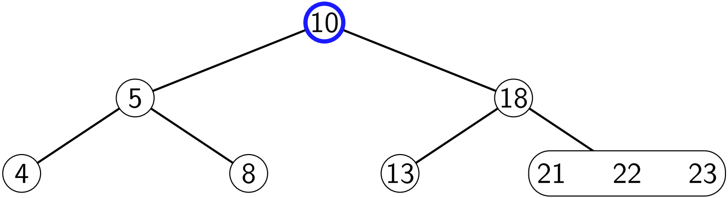

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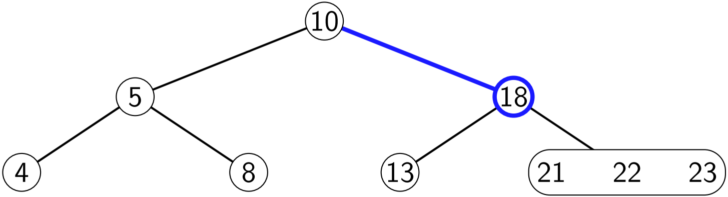

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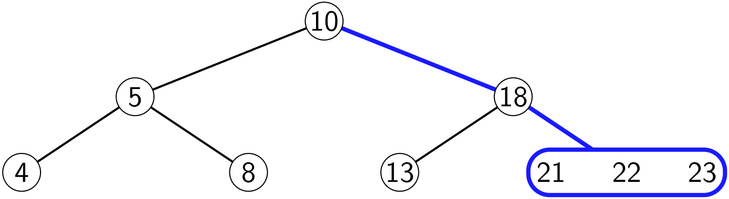

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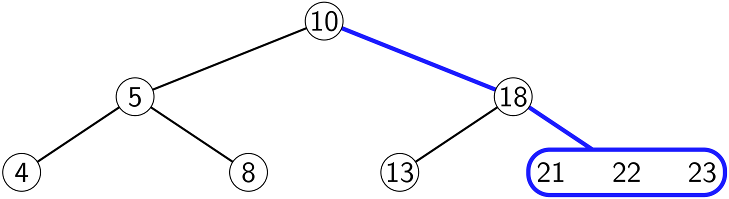

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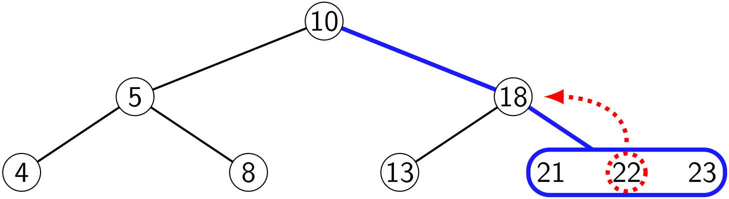

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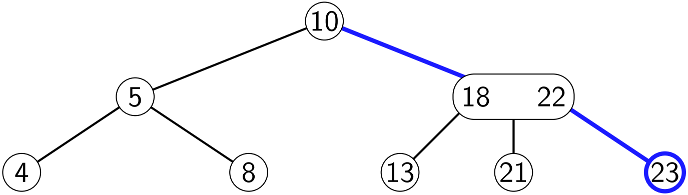

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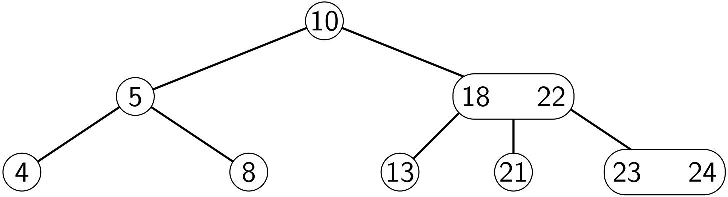

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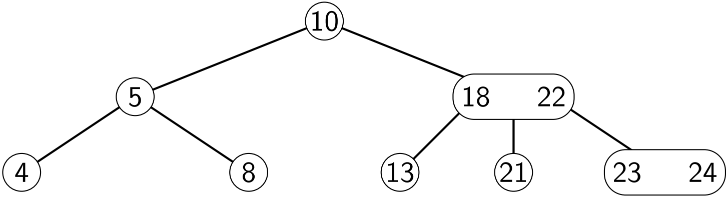

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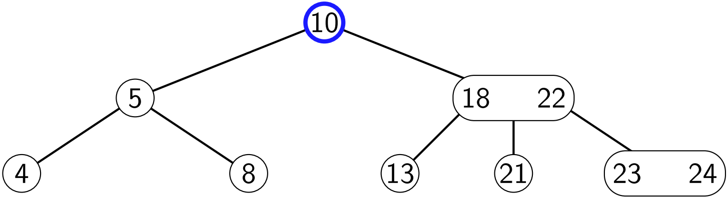

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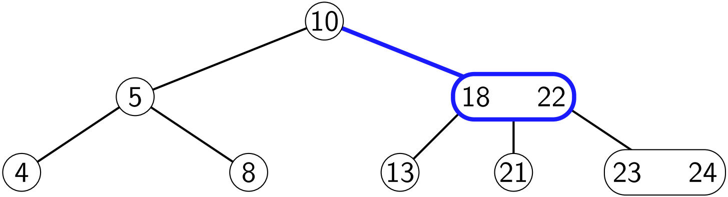

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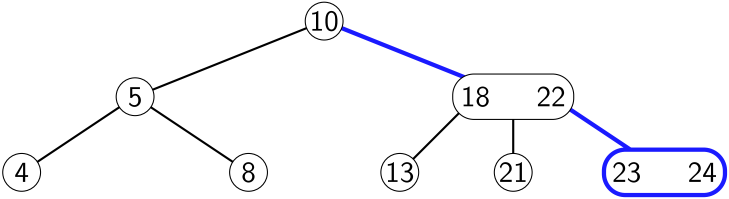

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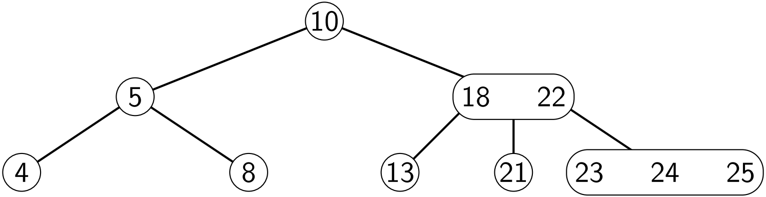

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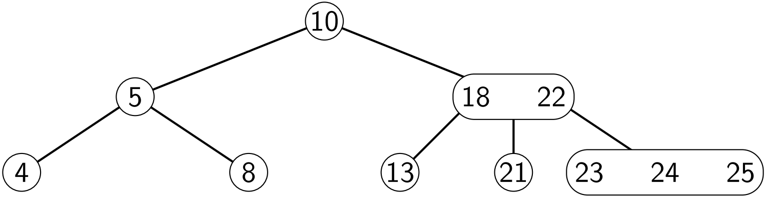

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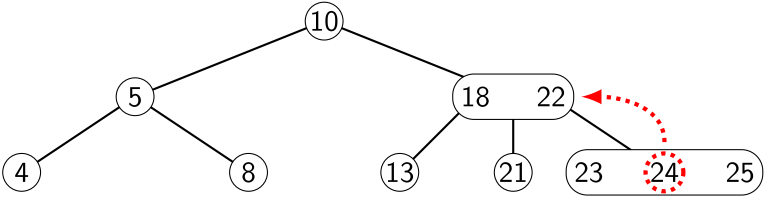

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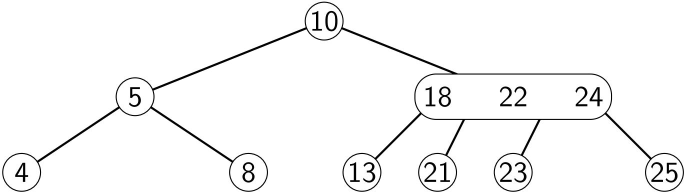

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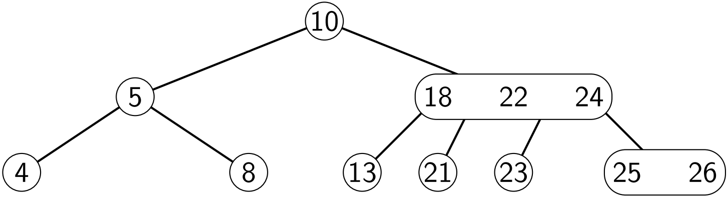

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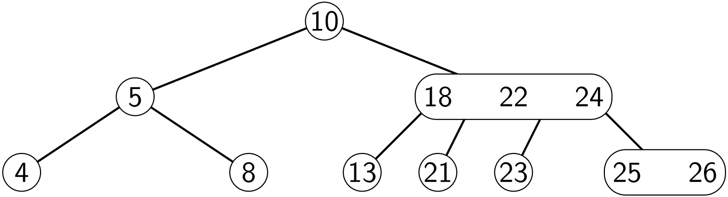
Inserting new values

Insert 27;



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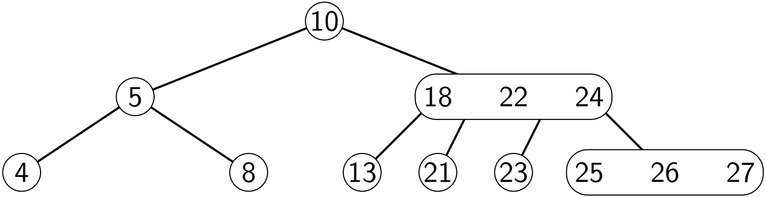

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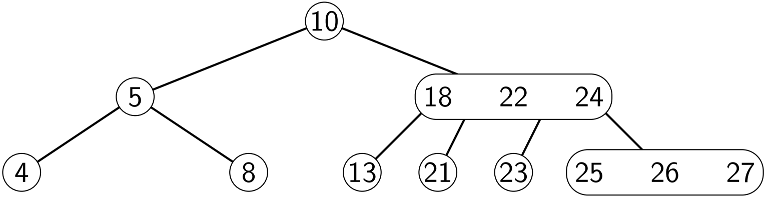

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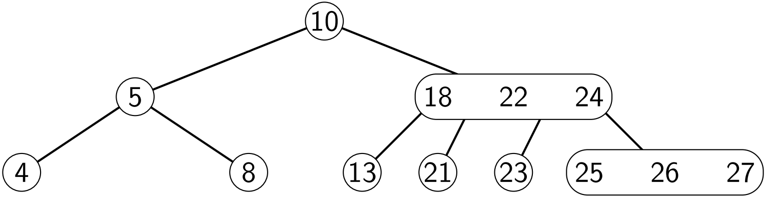

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If its parent is a 4-node as well, we're in trouble...

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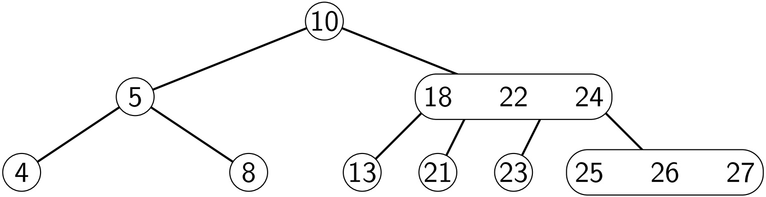

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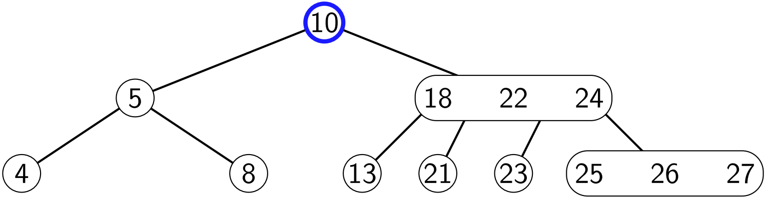

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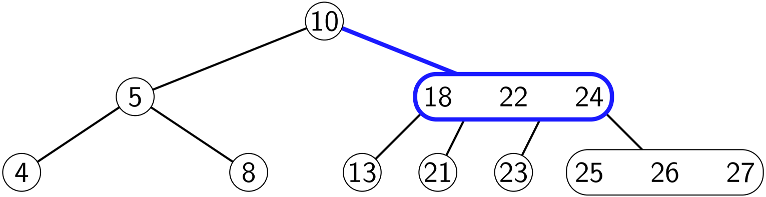

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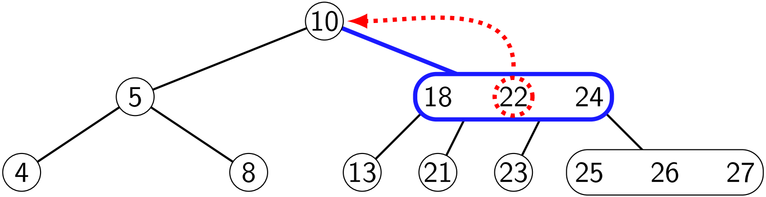

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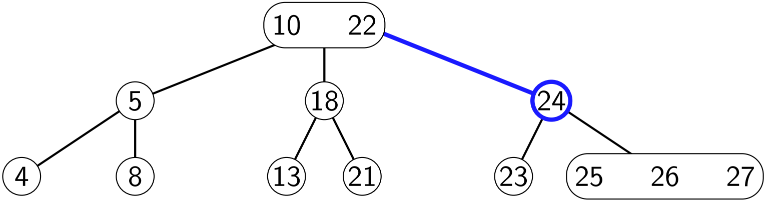

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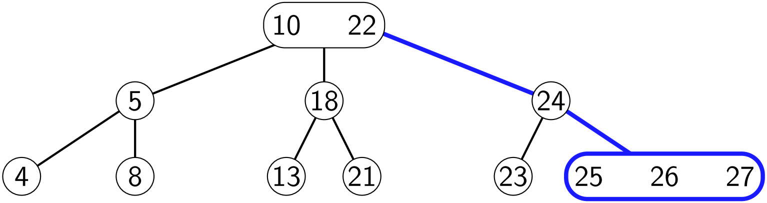

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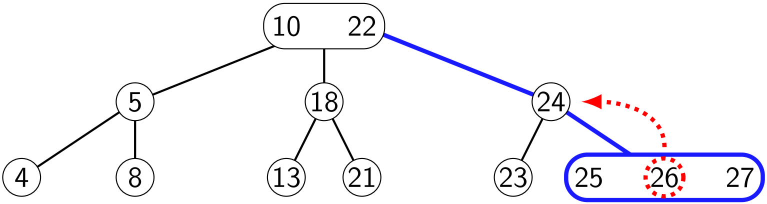

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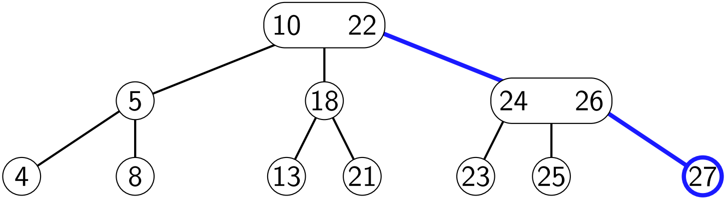

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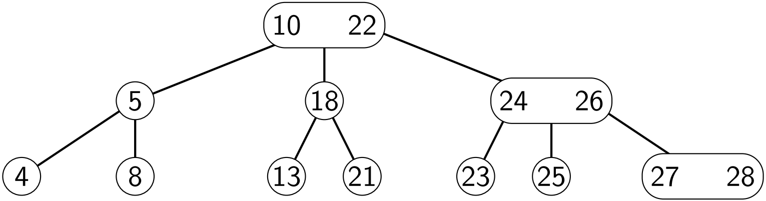

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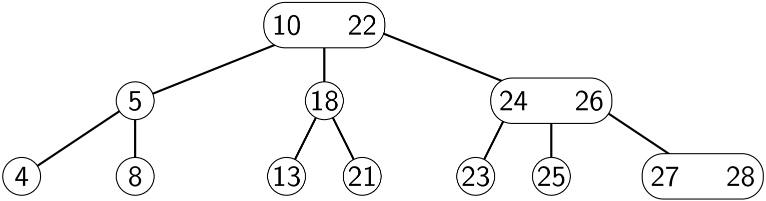

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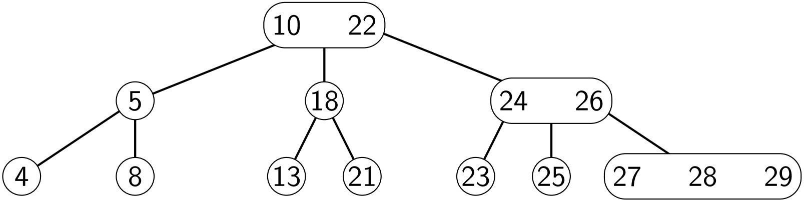

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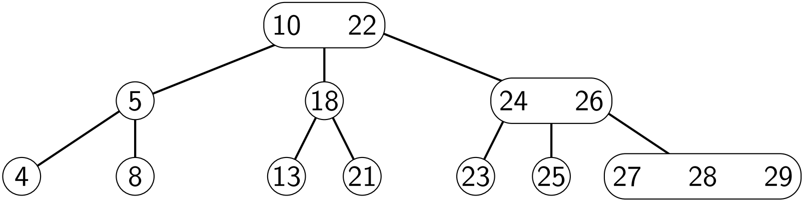

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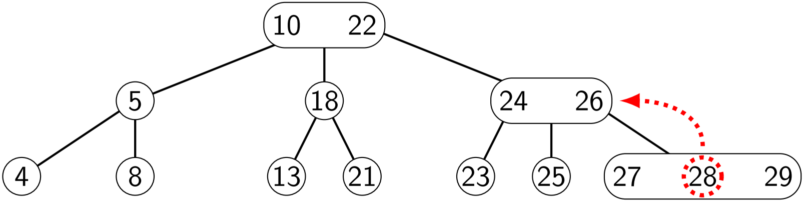

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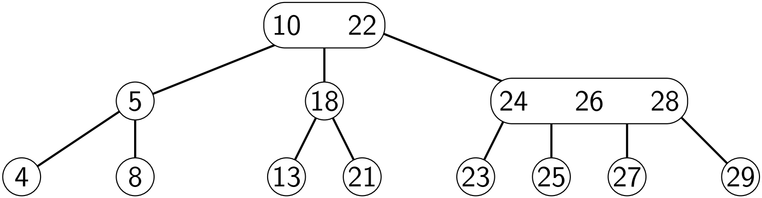

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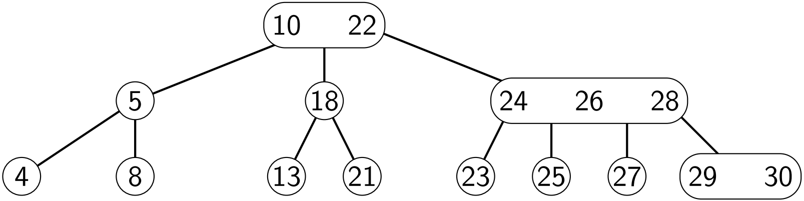

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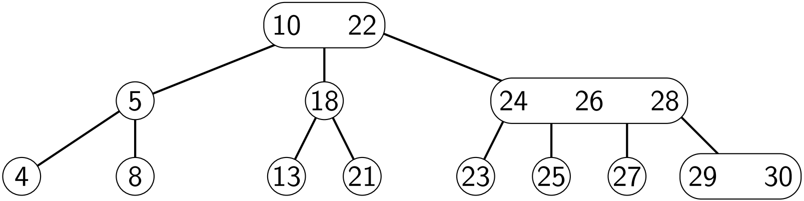

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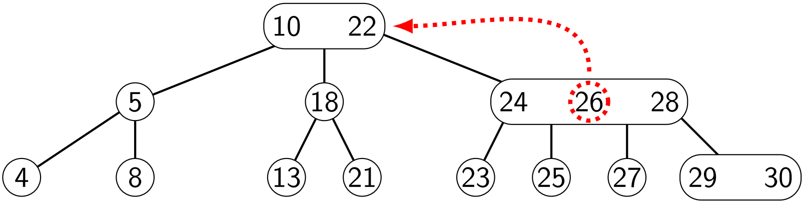

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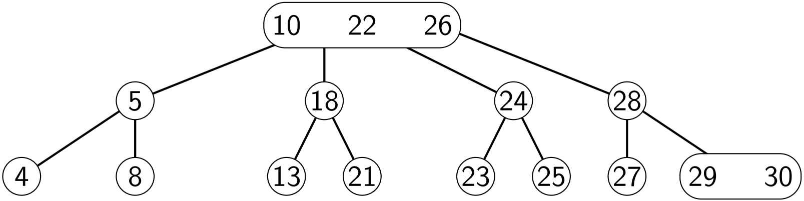

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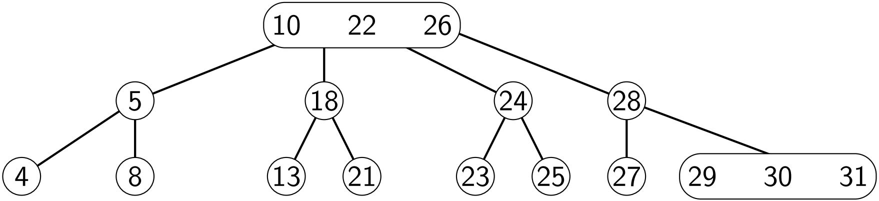

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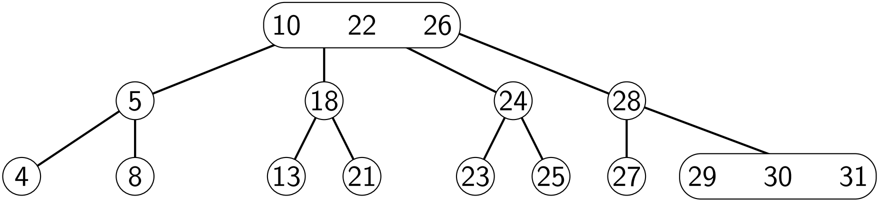

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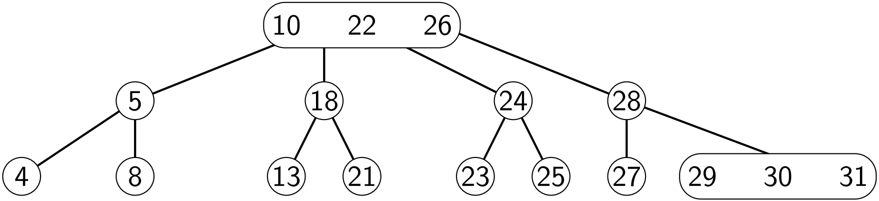

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If we have to split the root, d increases by 1.

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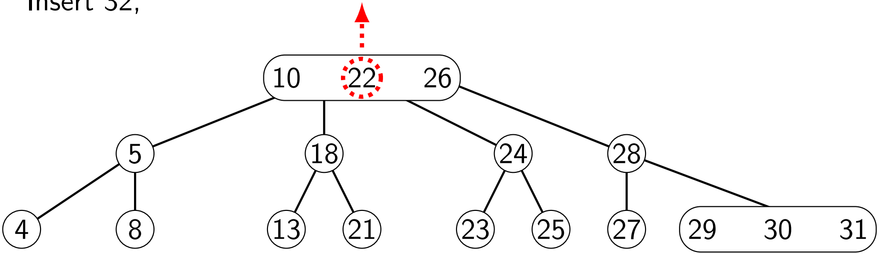

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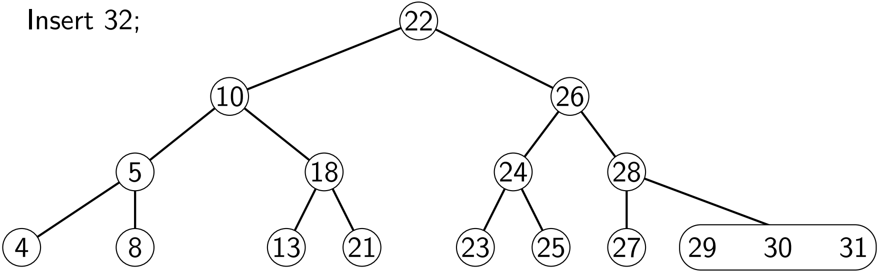

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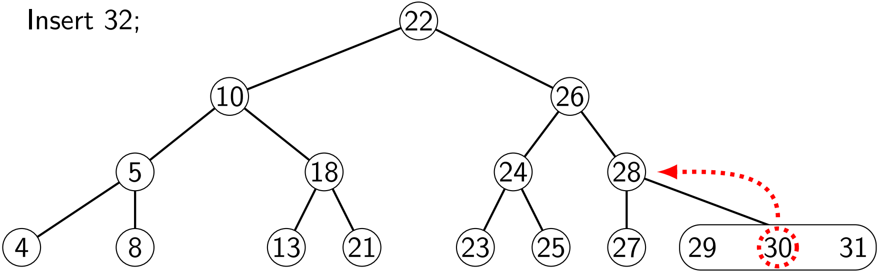

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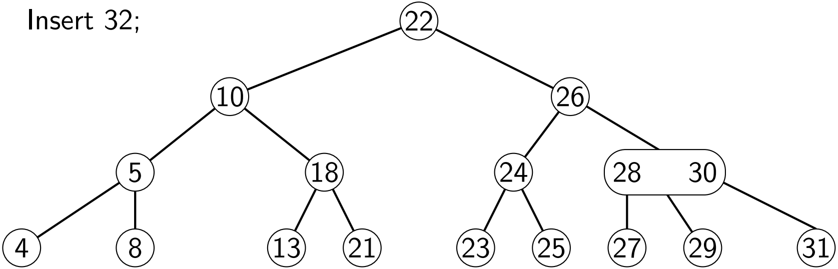

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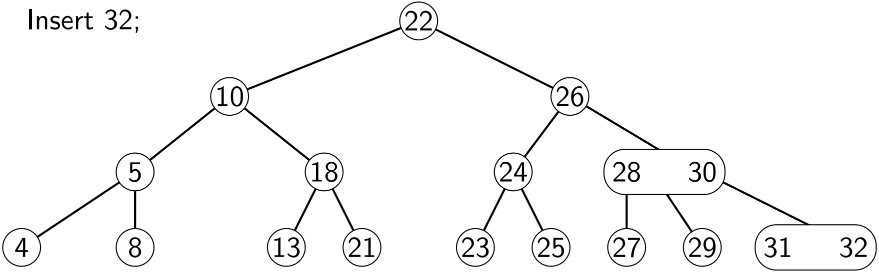

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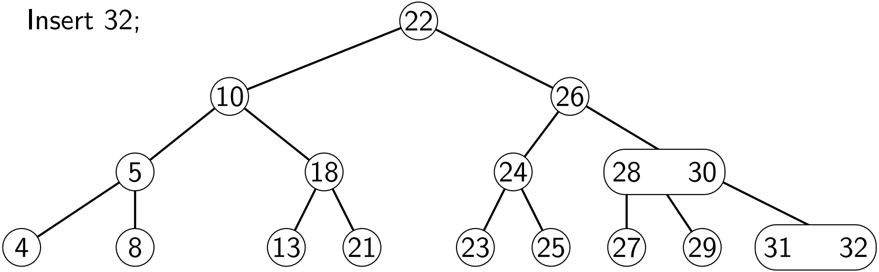

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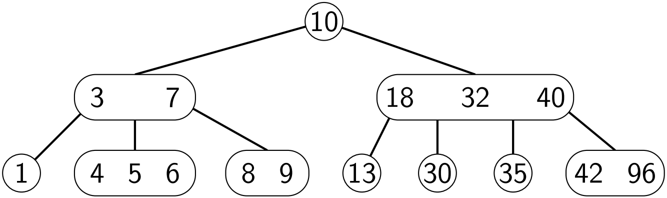
If we have to split the root, d increases by 1. But balance is maintained!

Summary of a 2-3-4 tree with distinct values (so far)



Finding a value v**:** Let x be the root. If $v \in x$, return a pointer to x. Otherwise, if x is a leaf, return `Not Found`. Otherwise, let k be such that x is a k-node, let $x_1 \leq \cdots \leq x_{k-1}$ be the values in x, let $x_0 = -\infty$, and let $x_k = \infty$; then $x_{i-1} < v < x_i$ for some i. Let c be the i'th child of x. Then repeat the process from the start, taking $x = c$.

Inserting a value v**:** First attempt to find v as above, **split**ting any 4-nodes encountered (including the root). After reaching a leaf L, and splitting it if it is a 4-node, add v to L.

Deleting a value v**:** Next time!