How the simplex algorithm works
COMS20010 (Algorithms II)

John Lapinskas, University of Bristol

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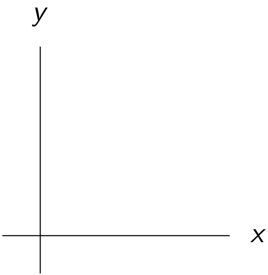
$x + y \to \max$ subject to



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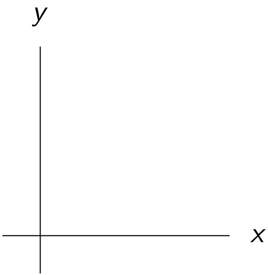

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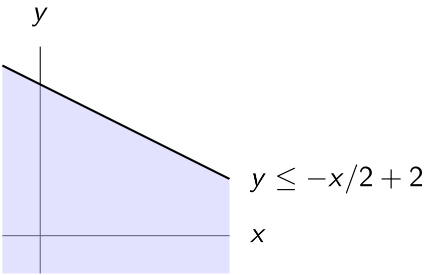

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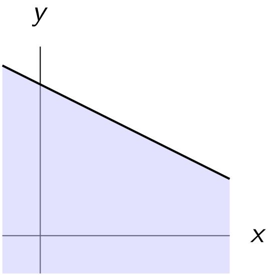

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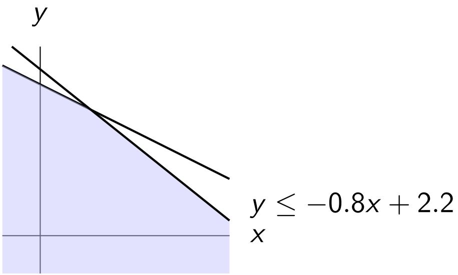

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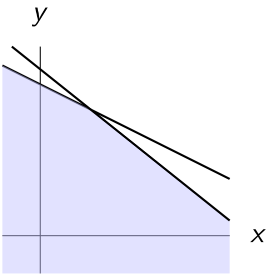

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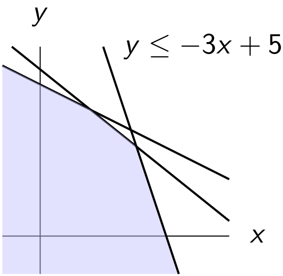

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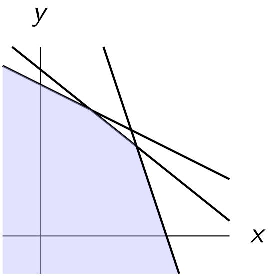

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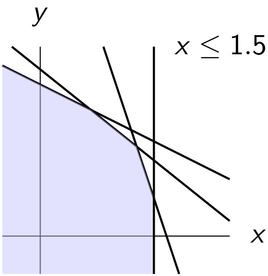

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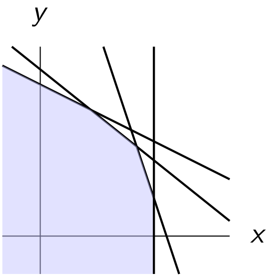

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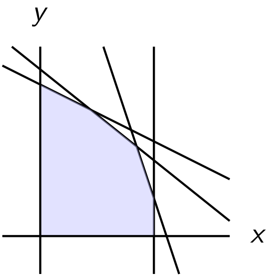

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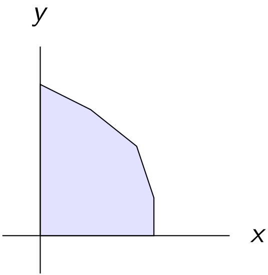

Let f be the objective function. Then for all c, $\{\vec{x} \in \mathbb{R}^n : f(\vec{x}) = c\}$ is an $(n-1)$-dimensional hyperplane in $\mathbb{R}^n$. The problem reduces to: how large can we take c and still have this hyperplane intersect the feasible polytope?

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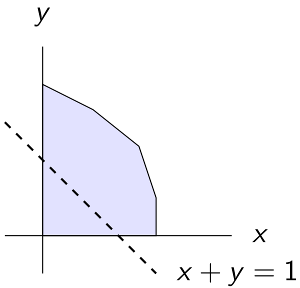

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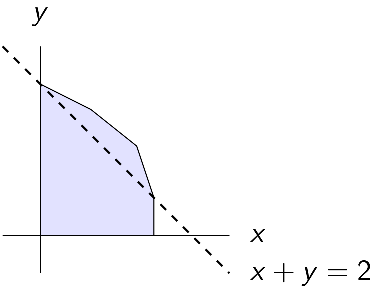

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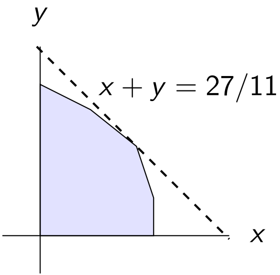

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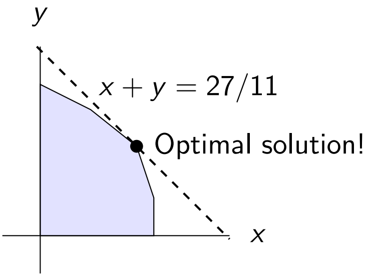

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The simplex algorithm

The n-variable constraints of an LP describe a feasible polytope in $\mathbb{R}^n$.

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Problem: There are often $\Omega(2^n)$ vertices, e.g. with a hypercube!

Running time of the simplex algorithm

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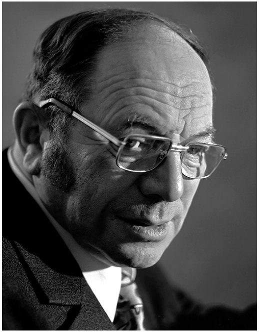
There are also **interior point algorithms**, which have a polynomial worst-case run-time, but which generally work less well in practice.

How not to do it...



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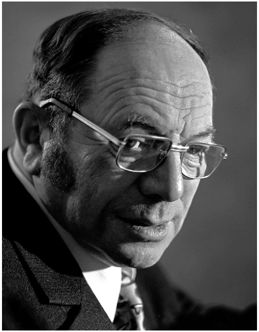

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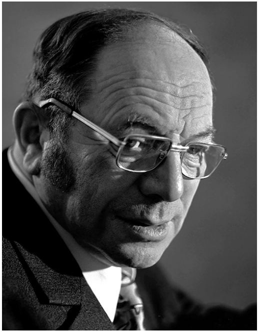

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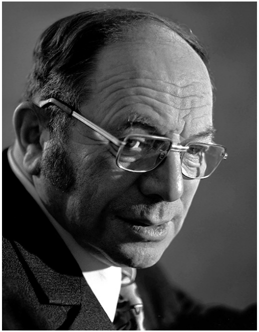

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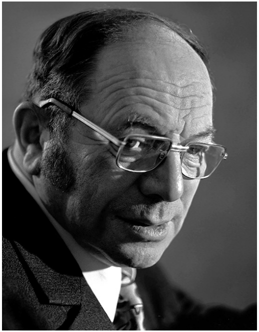

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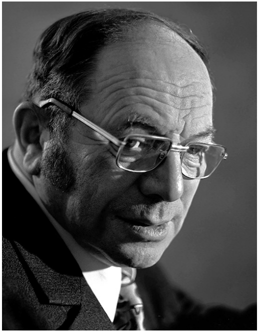

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- They had no accurate data.
- What should the objective function be?

A less lethal application: Approximation algorithms

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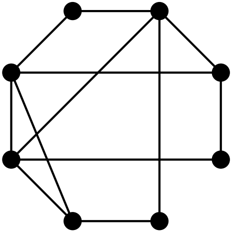

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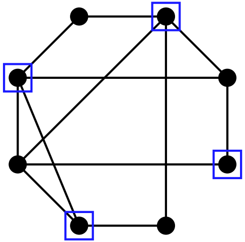

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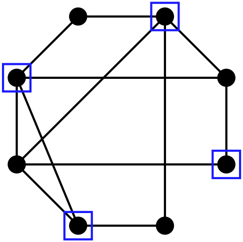

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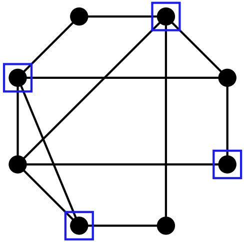

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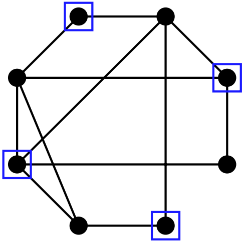

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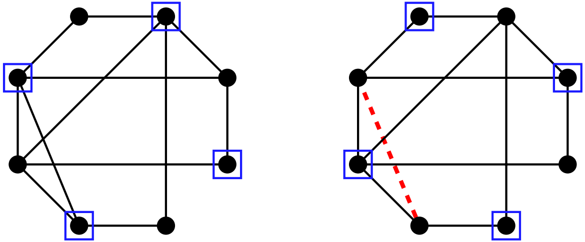

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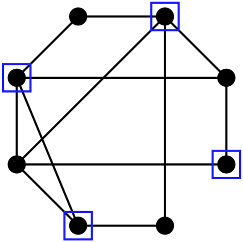

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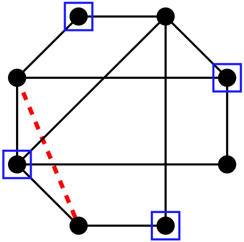

Not a valid vertex cover.

We would like to find the **smallest possible** vertex cover of G.

A **vertex cover** in a graph $G = (V, E)$ is a set $X \subseteq V$ such that every edge in E has at least one vertex in X.

We can express finding a minimum vertex cover as solving a linear program in which the solutions must be integers: an **integer linear program**.

Given a graph $G = (V, E)$, we assign a variable $x_v \in \{0, 1\}$ to each vertex v. We interpret $x_v = 1$ as "v is in the cover", and $x_v = 0$ as "v is not in the cover". We can then formulate the problem as:

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$x_u + x_v \geq 1$ for all $\{u, v\} \in E$;	$u \in X$ or $v \in X$ (or both) for all $\{u, v\} \in E$

A **vertex cover** in a graph $G = (V, E)$ is a set $X \subseteq V$ such that every edge in E has at least one vertex in X.

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Given a graph $G = (V, E)$, we assign a variable $x_v \in \{0, 1\}$ to each vertex v. We interpret $x_v = 1$ as "v is in the cover", and $x_v = 0$ as "v is not in the cover". We can then formulate the problem as:

$$\begin{aligned} \textstyle\sum_i x_i &\to \min \text{ subject to} & \text{Minimise } |X| \text{ subject to} \\ x_u + x_v &\geq 1 \text{ for all } \{u, v\} \in E; & u \in X \text{ or } v \in X \text{ (or both)} \\ & & \text{for all } \{u, v\} \in E \\ x_v &\leq 1 \text{ for all } v \in V; & \\ x_v &\geq 0 \text{ for all } v \in V; & [\text{Ensures } x_v \in \{0, 1\} \text{ for all } v] \\ x_v &\in \mathbb{N} \text{ for all } v \in V. & \end{aligned}$$

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Minimise $|X|$ subject to

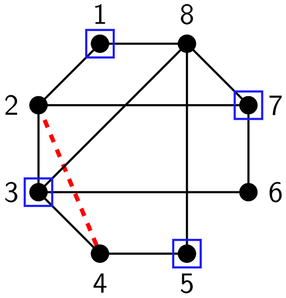
$u \in X$ or $v \in X$ (or both)

for all $\{u, v\} \in E$

[Ensures $x_v \in \{0, 1\}$ for all v]

Optimal solutions of this ILP correspond to minimum vertex covers of G, and minimum vertex covers of G correspond to optimal solutions.

An example of the ILP formulation of vertex cover



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$$x_u + x_v \geq 1 \text{ for all } \{u, v\} \in E;$$
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$$x_v \in \mathbb{N} \text{ for all } v \in V.$$

$X = \{1, 3, 5, 7\}$ is **not** a vertex cover.

Here we have $x_1 = x_3 = x_5 = x_7 = 1$ and $x_0 = x_2 = x_4 = x_6 = 0$.

The uncovered edge $\{2, 4\}$ corresponds to the constraint $x_2 + x_4 \geq 1$, which is violated.

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It's not hard to show (see problem sheet) that if a minimum vertex cover has size k, then X is indeed a vertex cover and $k \leq |X| \leq 2k$. So even though the problem is hard, we can still find an **approximate** solution!