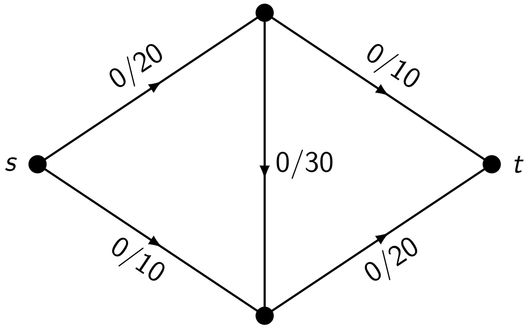
The Ford-Fulkerson algorithm
COMS20010 (Algorithms II)

John Lapinskas, University of Bristol

Now the definition of value is sorted out, how do we solve the problem?

How about a greedy approach? Repeatedly find paths from s to t with unused capacity and "push" more flow down them.



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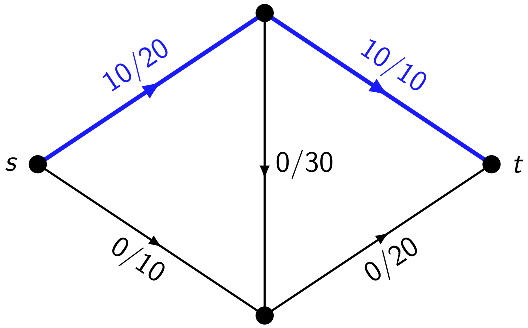

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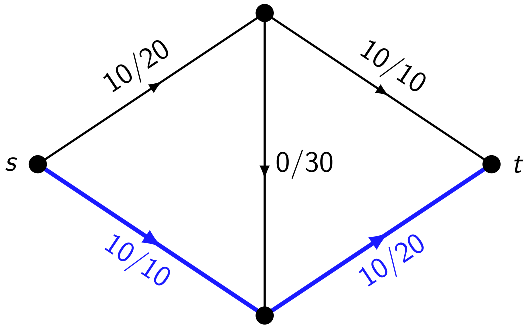

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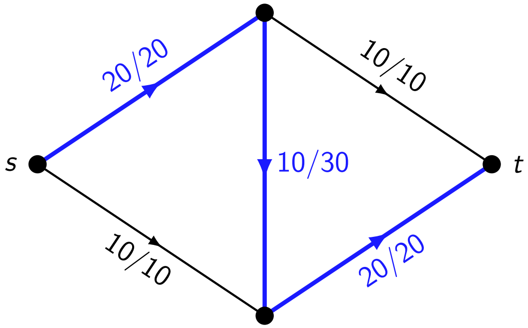
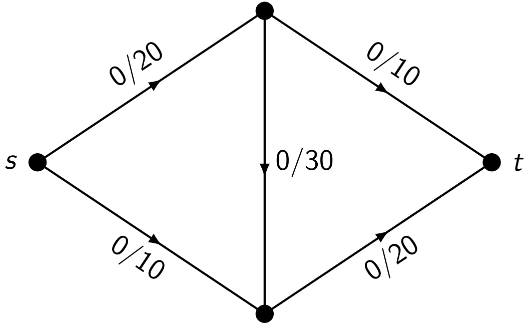


This flow has value $20 + 10 = 30$, which is best possible.
So a greedy approach can work... but it can also fail.

The problem: Find a **maximum flow**: a flow f maximising $v(f)$.

What if we'd chosen this path first instead?



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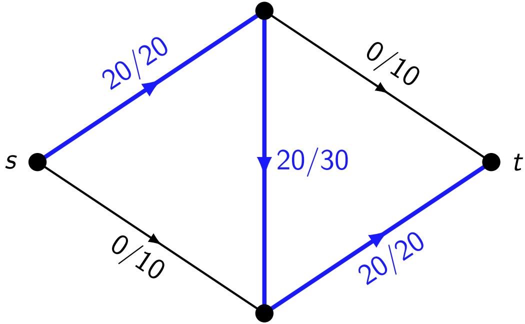

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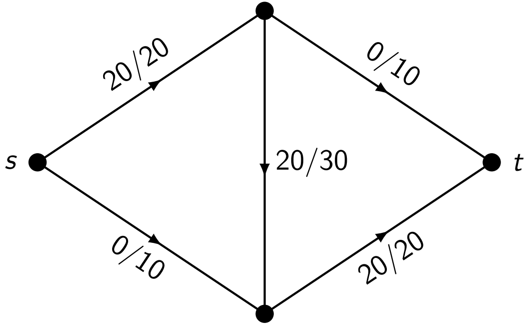
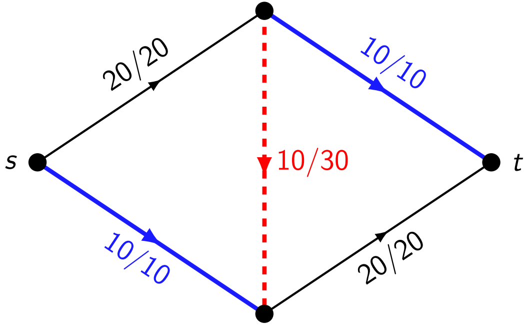


Now there are no more paths from s to t with spare capacity, but our flow only has value 20...

What if we allow ourselves to push flow *backwards* along a path?

What if we'd chosen this path first instead?



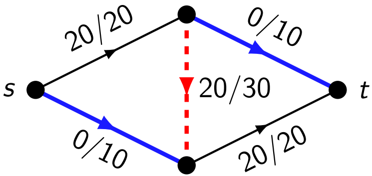
Now there are no more paths from s to t with spare capacity, but our flow only has value 20...

What if we allow ourselves to push flow *backwards* along a path?
We get a maximum flow! Now to generalise this...

A **flow** is a function $f\colon E \to \mathbb{R}$ such that for all $e \in E$ and $v \in V \setminus \{s,t\}$:

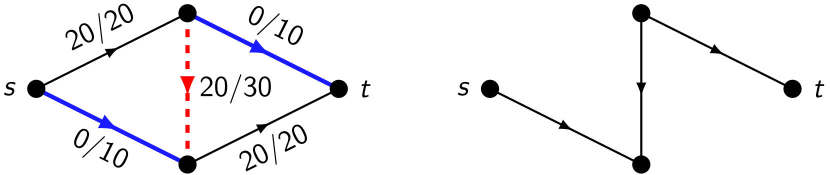
- $0 \le f(e) \le c(e)$;
- $f^+(v) := \sum_{w \in N^+(v)} f(v,w) = \sum_{u \in N^-(v)} f(u,v) =: f^-(v)$.

The problem: Find a **maximum flow**: a flow f maximising $v(f)$.



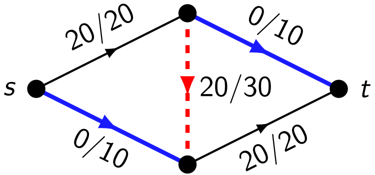
We want to say: an augmenting path for a flow f is an **undirected** path from s to t which we can push flow along. So forward edges e have $f(e) < c(e)$, and backward edges e have $f(e) > 0$.

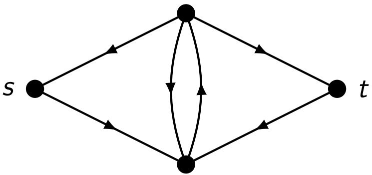
But there's an annoying technicality with bidirected edges...



We define the **residual graph** G_f of (G, c, s, t) on $V(G)$ as follows:

- If $(u, v) \in E(G)$ with $f(e) < c(e)$, add (u, v) to $E(G_f)$; call this a **forward edge**.





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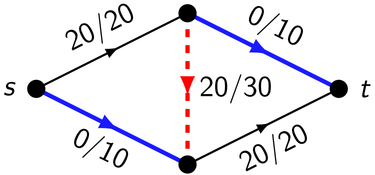

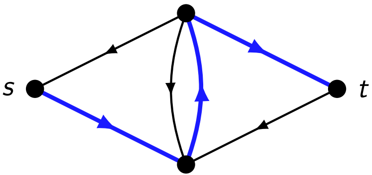

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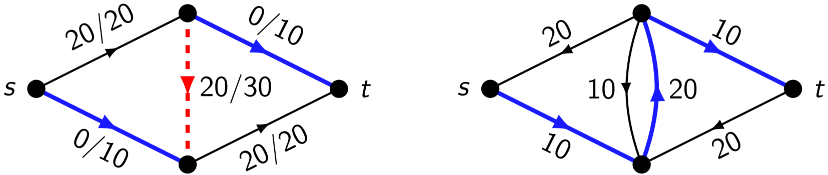
An **augmenting path** P is a directed path from s to t in G_f.



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The **residual capacity** of (u, v) in G_f is $\max\{c(u, v) - f(u, v), f(v, u)\}$.

The **residual capacity** of P is the minimum residual capacity of its edges. (This is the amount of flow we can push through P.)

$(u,v) \in E(G)$ with $f(e) < c(e)$ yields a **forward edge** $(u,v) \in E(G_f)$.
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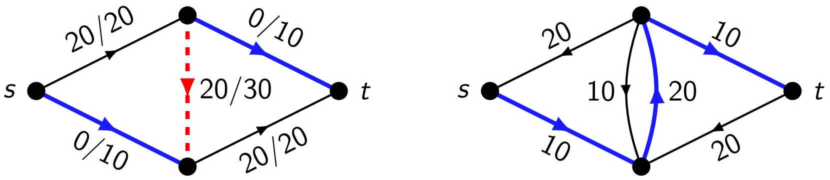

Define $\texttt{Push}(G, c, s, t, f, P)$ as follows:

- Let C be the residual capacity of P. (Here C is 10.)
- For each edge (u,v) of P: if $c(u,v) - f(u,v) \geq C$, then add C to $f(u,v)$; otherwise, we have $f(v,u) \geq C$, so subtract C from $f(v,u)$.

$(u, v) \in E(G)$ with $f(e) < c(e)$ yields a **forward edge** $(u, v) \in E(G_f)$.
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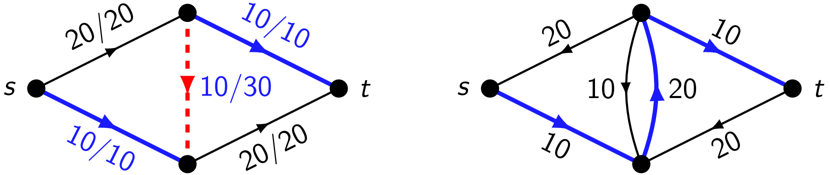

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Lemma 3: $\mathtt{Push}(G, c, s, t, f, P)$ returns a new flow f', with value $v(f') = v(f) + C$, in $O(|V(G)|)$ time. □

The Ford-Fulkerson Algorithm

Algorithm: FordFulkerson

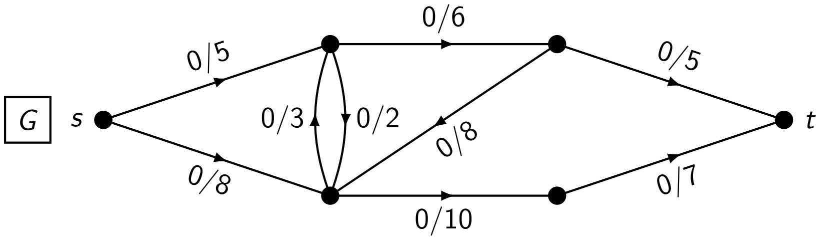
Input : A (weakly connected) flow network (G, c, s, t).
Output : A flow f with no augmenting paths.

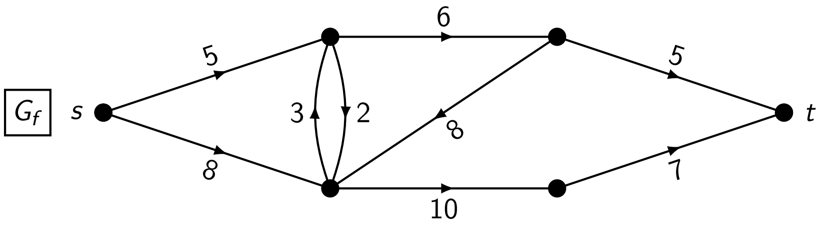
```
1 begin
2     Construct the flow f with f(e) = 0 for all e ∈ E(G).
3     Construct the residual graph G_f.
4     while G_f contains a path P from s to t do
5         Find P using depth-first (or breadth-first) search.
6         Update f ← Push(G, c, s, t, f, P).
7         Update G_f on the edges of P.
8     Return f.
```

By Lemma 3, every iteration of 4–7 increases $v(f)$ by at least 1. So if f^* is a maximum flow, there are at most $v(f^*)$ iterations in total.

Every step takes $O(|E|)$ time or $O(|V|)$ time, and since G is weakly connected we have $|V| = O(|E|)$. So the running time is $O(v(f^*)|E|)$.

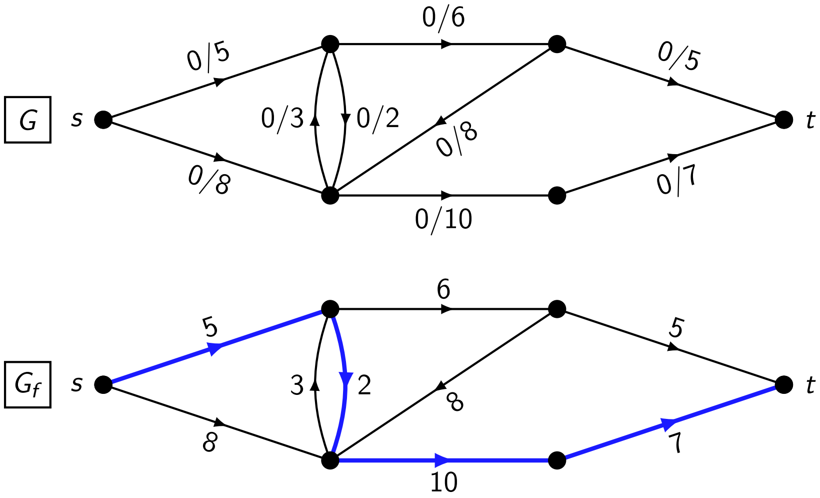
Worked example





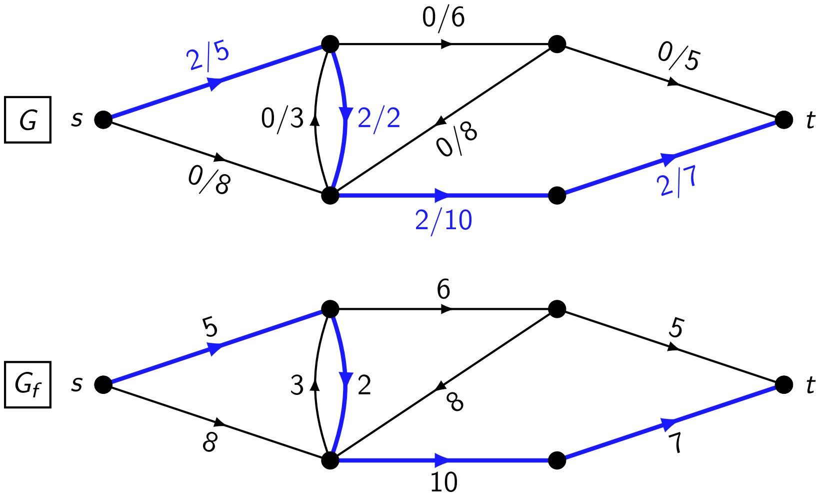
Initialise flow and construct G_f.

Worked example



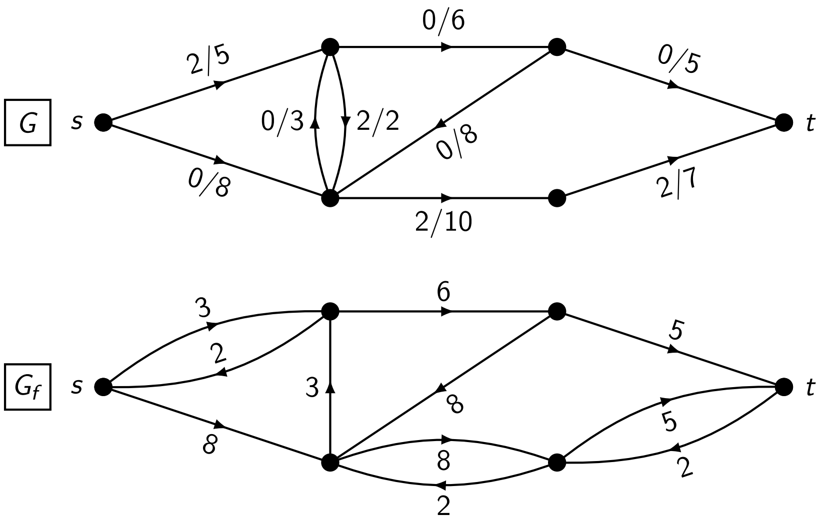
Apply depth-first search to find an augmenting path in G_f.

Worked example



Push flow along the path. (This path has residual capacity 2.)

Worked example



Update G_f along the path.

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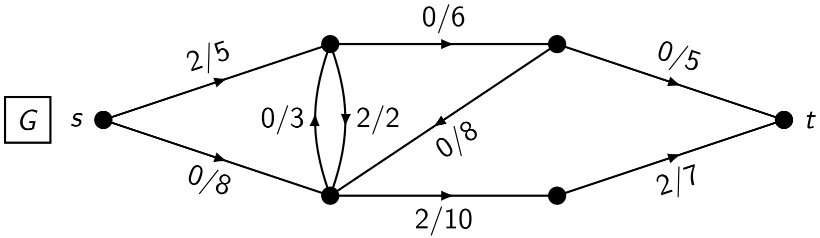

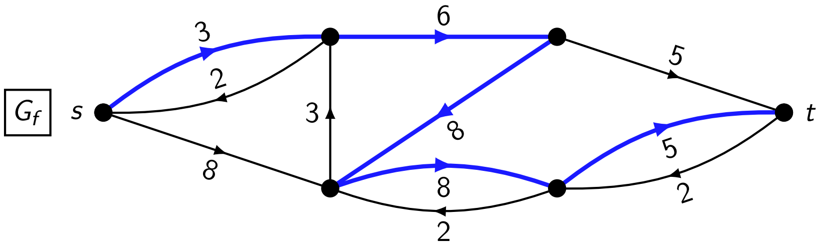

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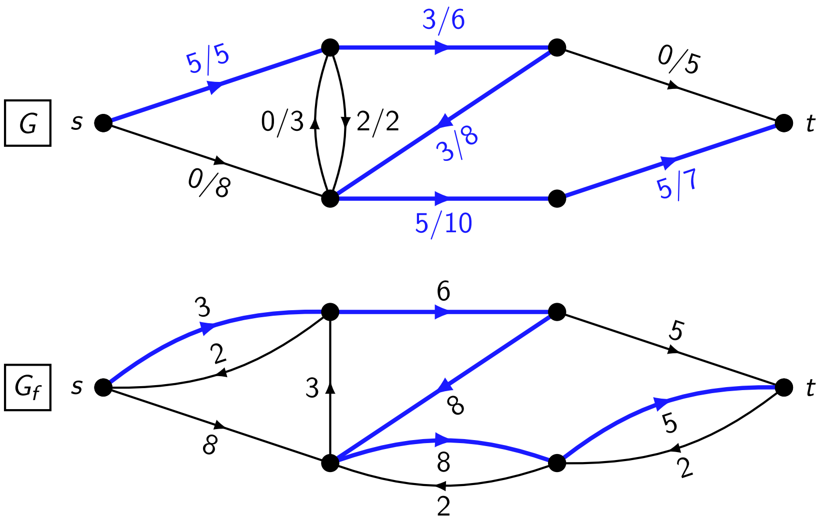

Push flow along the path. (This path has residual capacity 3.)

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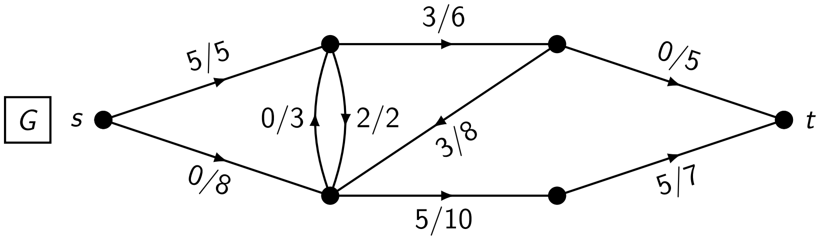

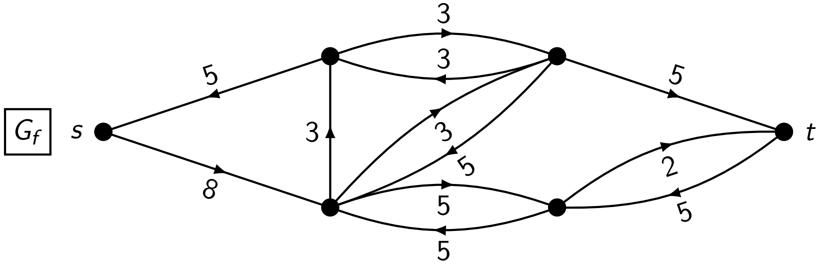

Update G_f along the path.

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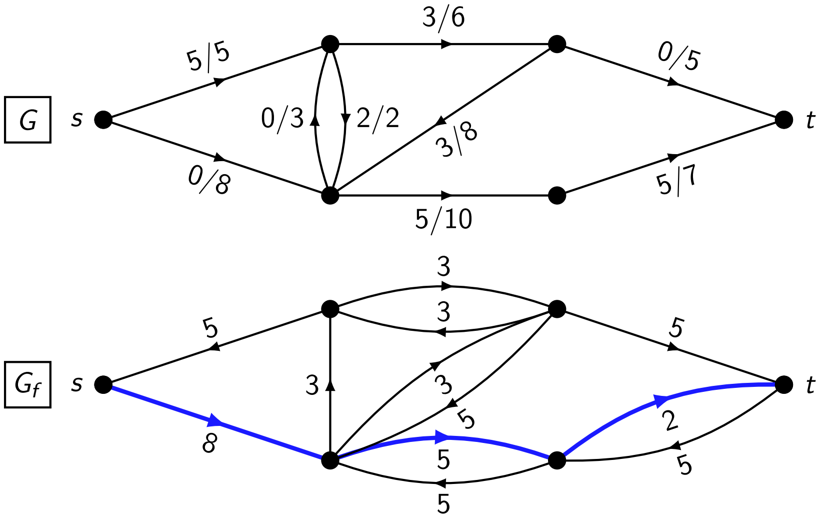

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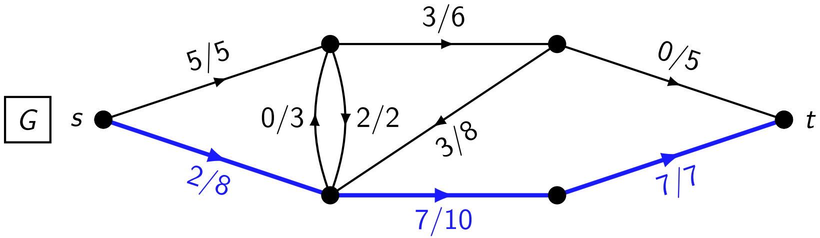
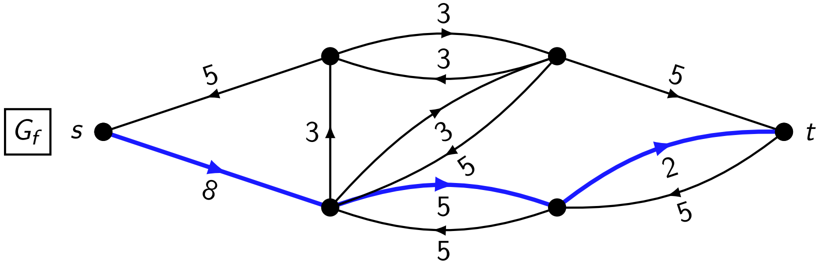
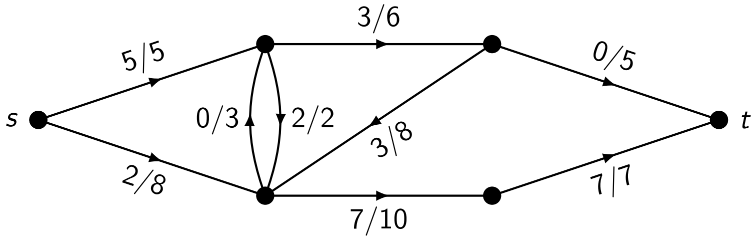


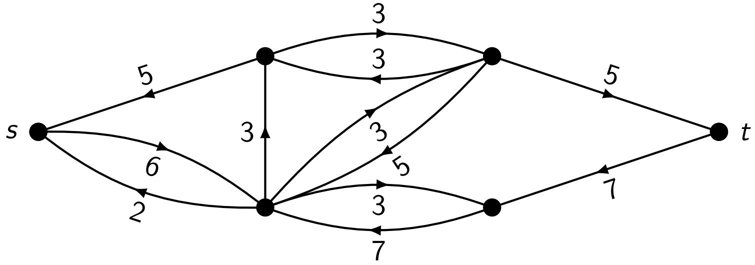


Push flow along the path. (This path has residual capacity 2.)

$\boxed{G}$



$\boxed{G_f}$



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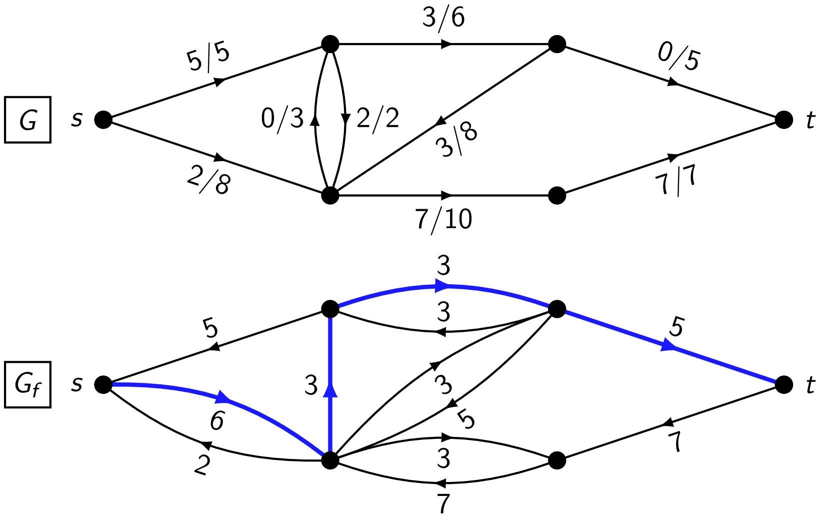
Worked example



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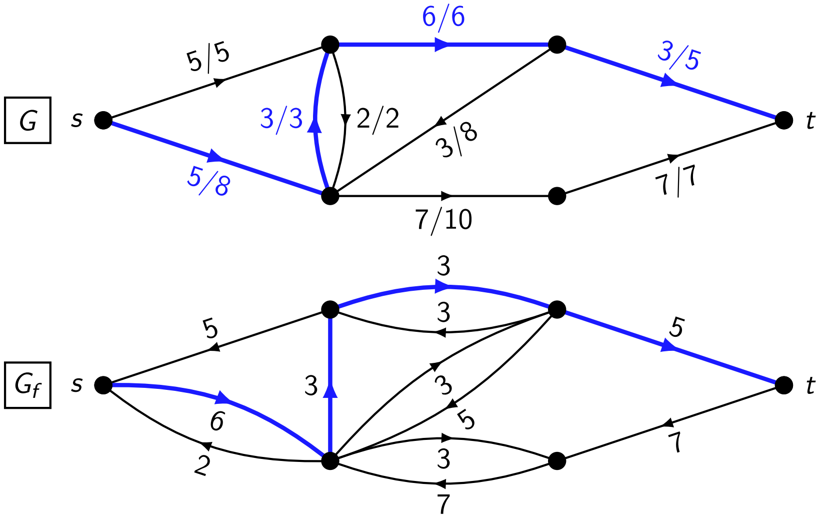

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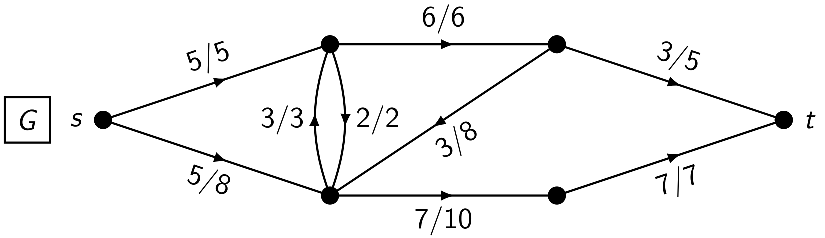

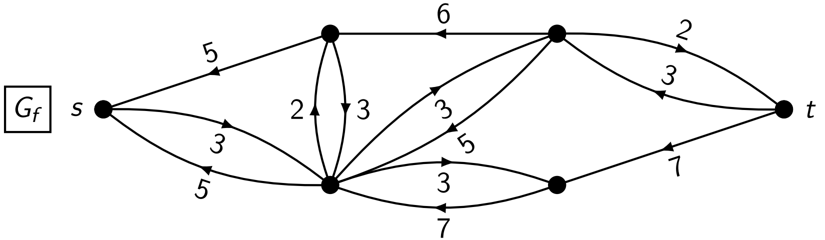

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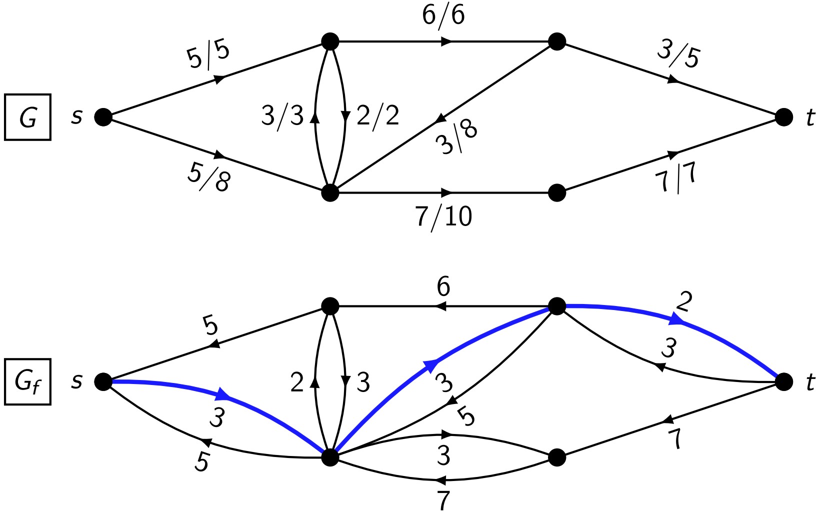

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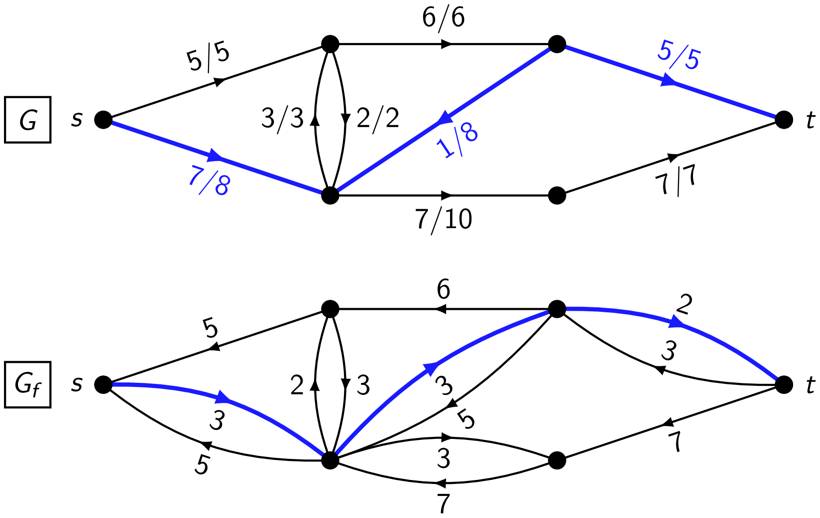

Push flow along the path. (This path has residual capacity 2.)

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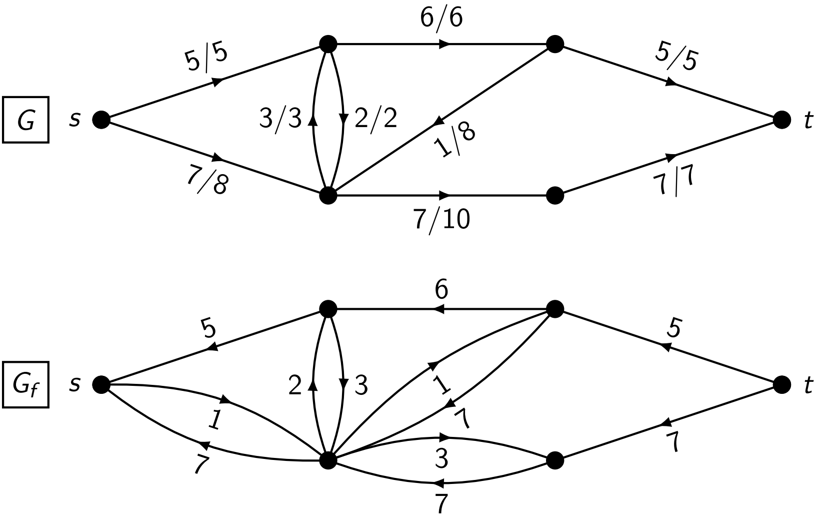

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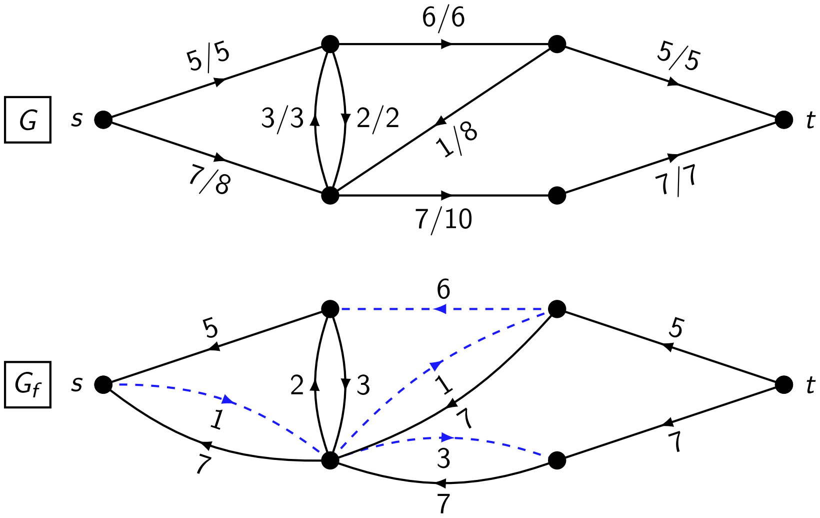

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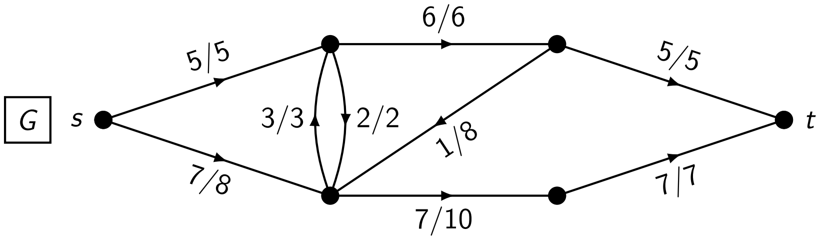
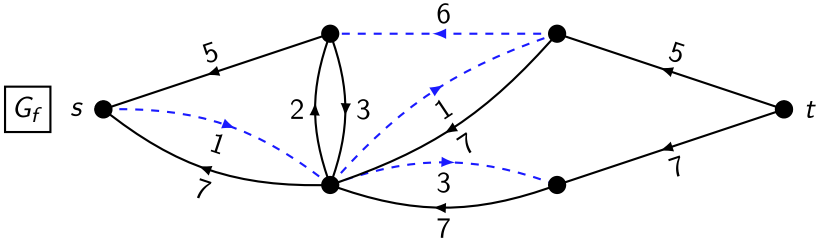


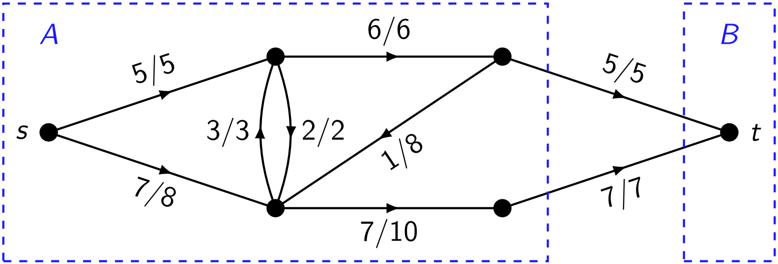


No such path exists, so we're done! This flow has value $5 + 7 = 12$.

Why does this work?

A **cut** is any pair of disjoint sets $A, B \subseteq V$ with $A \cup B = V$, $s \in A$ and $t \in B$. (So A and B partition V, the source is in A and the sink is in B.)

Lemma 2: For all cuts (A, B), $v(f) = f^+(A) - f^-(A) = f^-(B) - f^+(B)$.

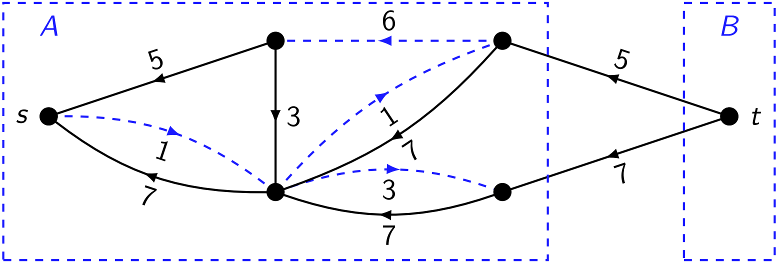


Write $c^+(A) = \sum_{e \text{ out of } A} c(e)$. By Lemma 2, **any** flow g has value $v(g) = g^+(A) - g^-(A) \le c^+(A) = 12$, and our output flow has value 12. So it must be maximum.

We can use the same argument to prove Ford-Fulkerson always works.

Lemma 2: For all cuts (A, B), $v(f) = f^+(A) - f^-(A) = f^-(B) - f^+(B)$.

To prove the flow f returned by Ford-Fulkerson is **always** maximum by this argument, we will show there is **always** a cut (A, B) with $v(f) = c^+(A)$, i.e. with $f^+(A) = c^+(A)$ and $f^-(A) = 0$.



We take $A = \{v \in V(G) \colon v \text{ reachable from } s \text{ in } G_f\}$, and $B = V(G) \setminus A$.

(A, B) is a cut: $s \in A$, and $t \notin A$ since f has no augmenting paths. ✓

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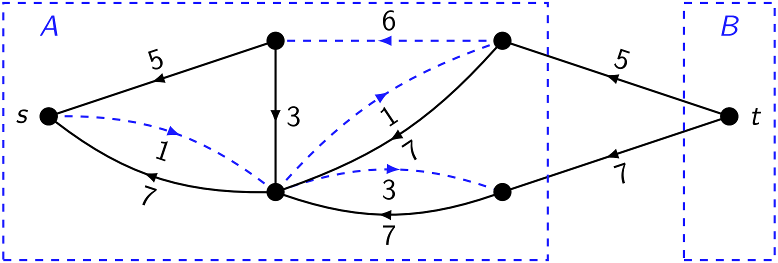

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(A, B) is a cut: ✓

$f^+(A) = c^+(A)$: No $A \to B$ forward edges in G_f ⇒ every $A \to B$ edge in G is filled to capacity ⇒ $f^+(A) = c^+(A)$. ✓

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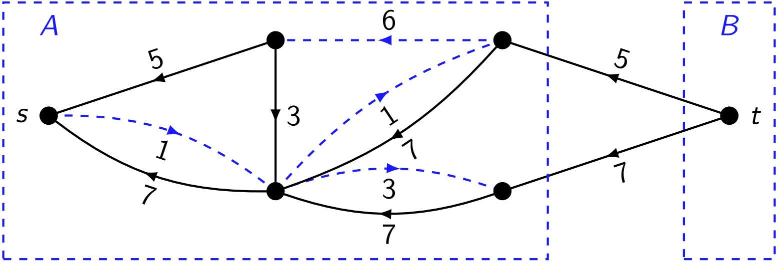

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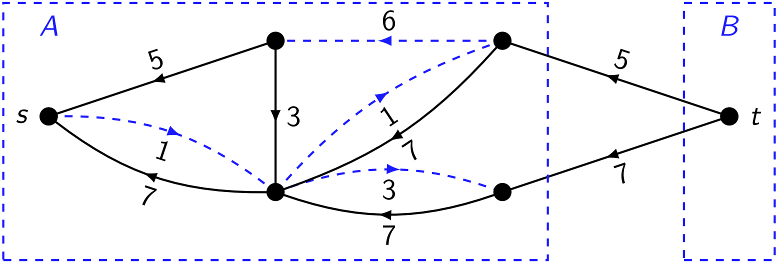
(A, B) is a cut: ✓ **$f^+(A) = c^+(A)$:** ✓

$f^-(A) = 0$: No $A \to B$ backward edges in G_f $\Rightarrow$ every $B \to A$ edge in G has zero flow $\Rightarrow f^-(A) = 0$. ✓

To prove the flow f returned by Ford-Fulkerson is **always** maximum by this argument, we will show there is **always** a cut (A, B) with $v(f) = c^+(A)$, i.e. with $f^+(A) = c^+(A)$ and $f^-(A) = 0$.



We take $A = \{v \in V(G) \colon v \text{ reachable from } s \text{ in } G_f\}$, and $B = V(G) \setminus A$.

$\boldsymbol{(A, B)}$ **is a cut:** ✓ $\boldsymbol{f^+(A) = c^+(A)}$**:** ✓ $\boldsymbol{f^-(A) = 0}$**:** ✓

So by Lemma 2, every other flow g has value $g^+(A) - g^-(A) \leq c^+(A) = v(f)$. Thus f is a maximum flow and Ford-Fulkerson is correct. □

We have proved three results for the price of one!

Theorem: Ford-Fulkerson always returns a maximum flow. □

Theorem: There is always a maximum flow with integer values.

Proof: The maximum flow returned by Ford-Fulkerson has this property. (We can prove this easily with a loop invariant: f starts with value zero, and each iteration of the main loop adds an integer to f's value.) □

Max-flow min-cut theorem: The value of a maximum flow is equal to the minimum capacity of a cut, i.e. the minimum value of $c^+(A)$ over all cuts (A, B).

Proof: Let f be a maximum flow, and let (A, B) be a cut minimising $c^+(A)$. We already proved $v(f) \leq c^+(A)$. Moreover, there is no augmenting path for f, so exactly as before, there is a cut (A', B') with $c^+(A') = v(f)$; thus $v(f) \geq c^+(A)$. The result follows. □