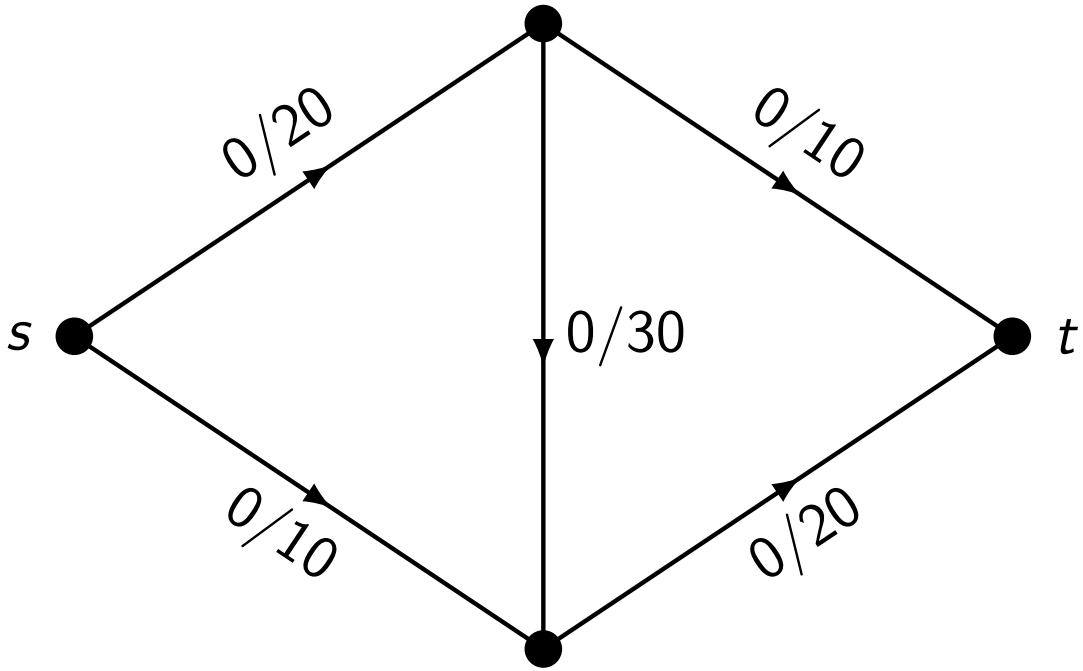
The Ford-Fulkerson algorithm
COMS20010 (Algorithms II)

John Lapinskas, University of Bristol

The problem: Find a **maximum flow**: a flow f maximising $v(f)$.

Now the definition of value is sorted out, how do we solve the problem?

How about a greedy approach? Repeatedly find paths from s to t with unused capacity and "push" more flow down them.



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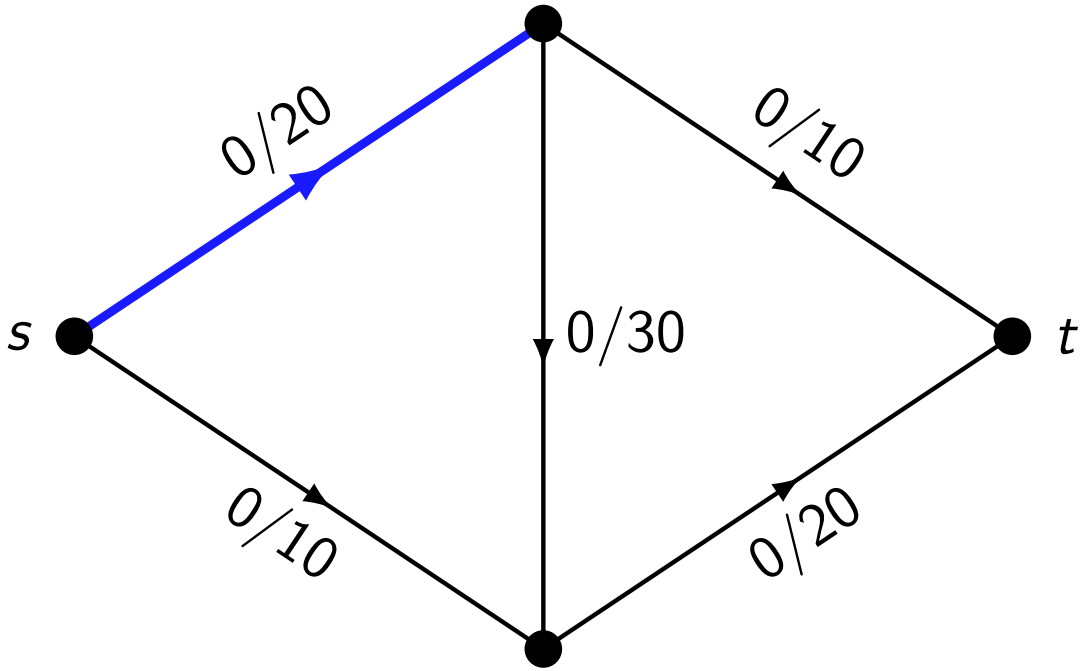

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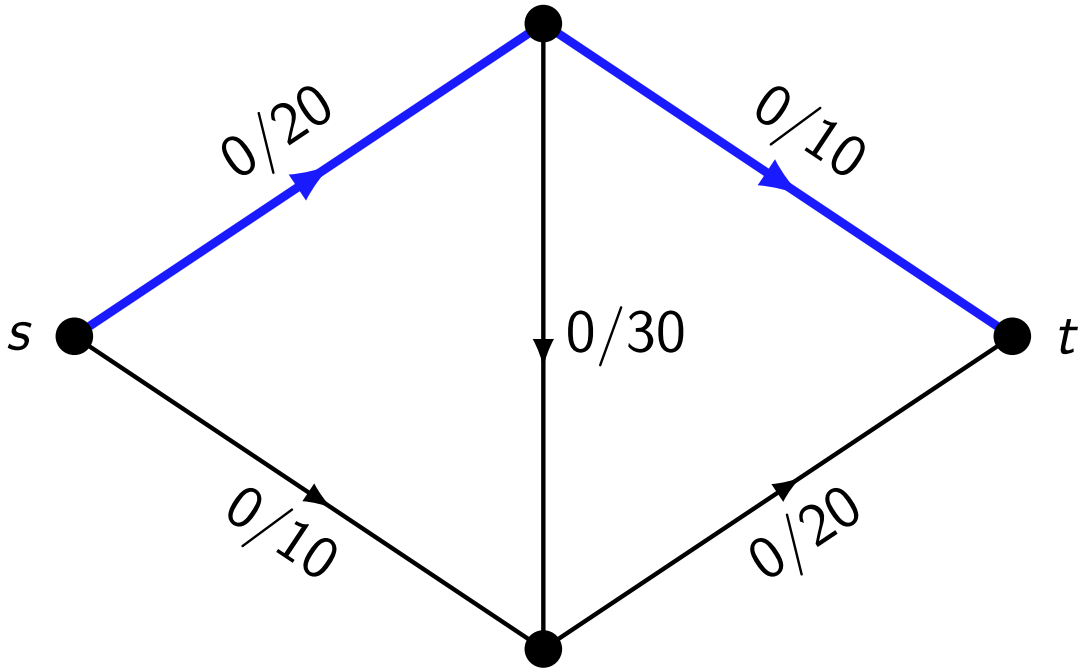

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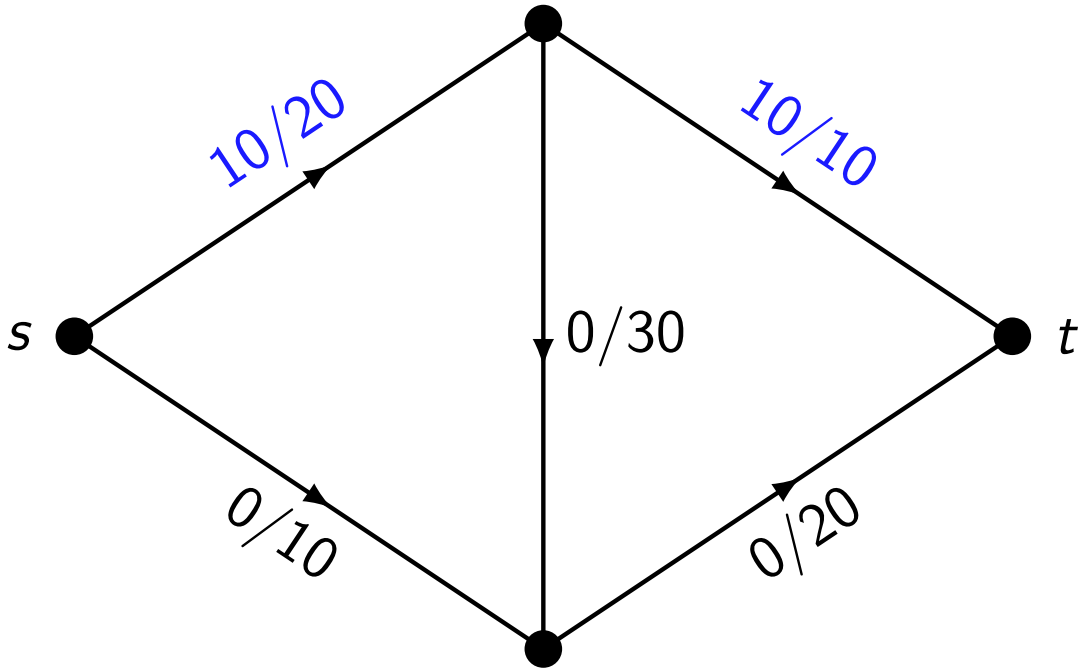

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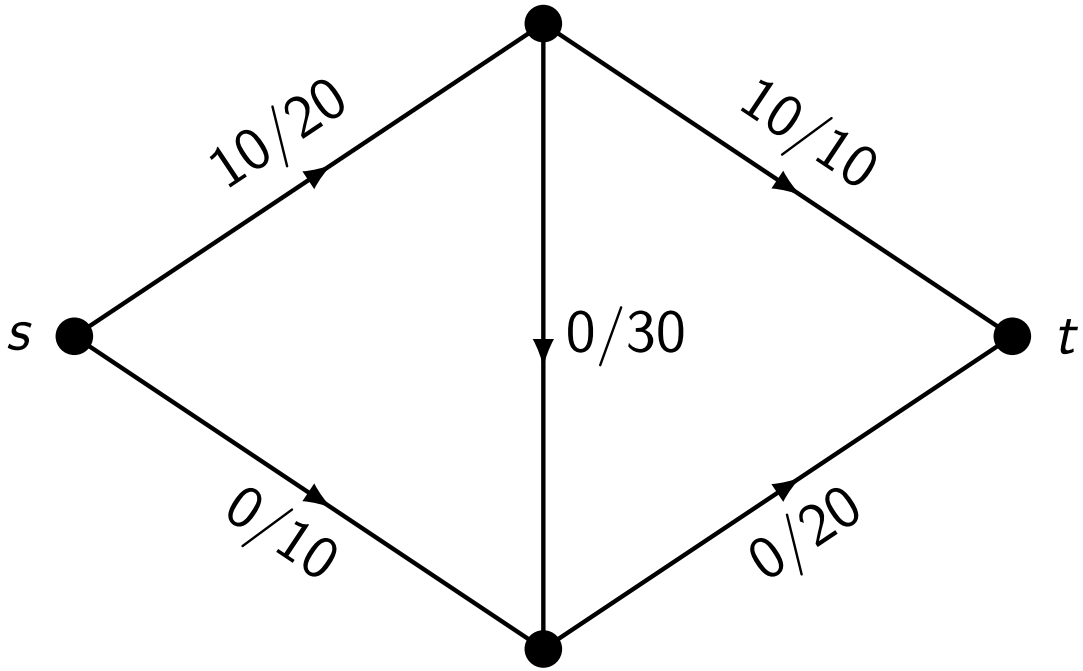

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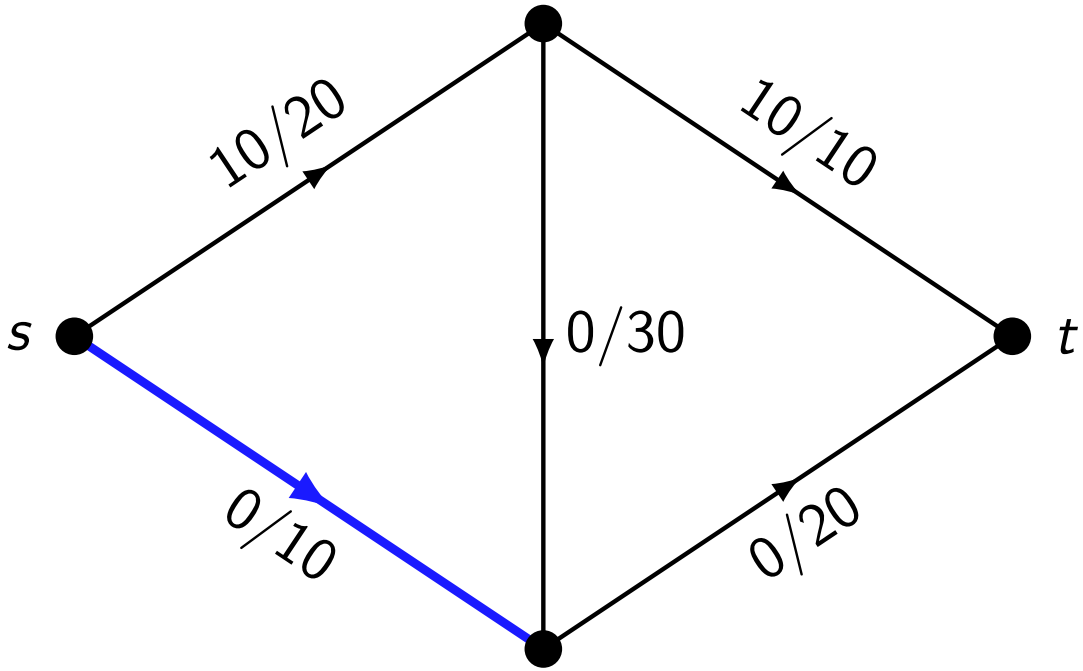

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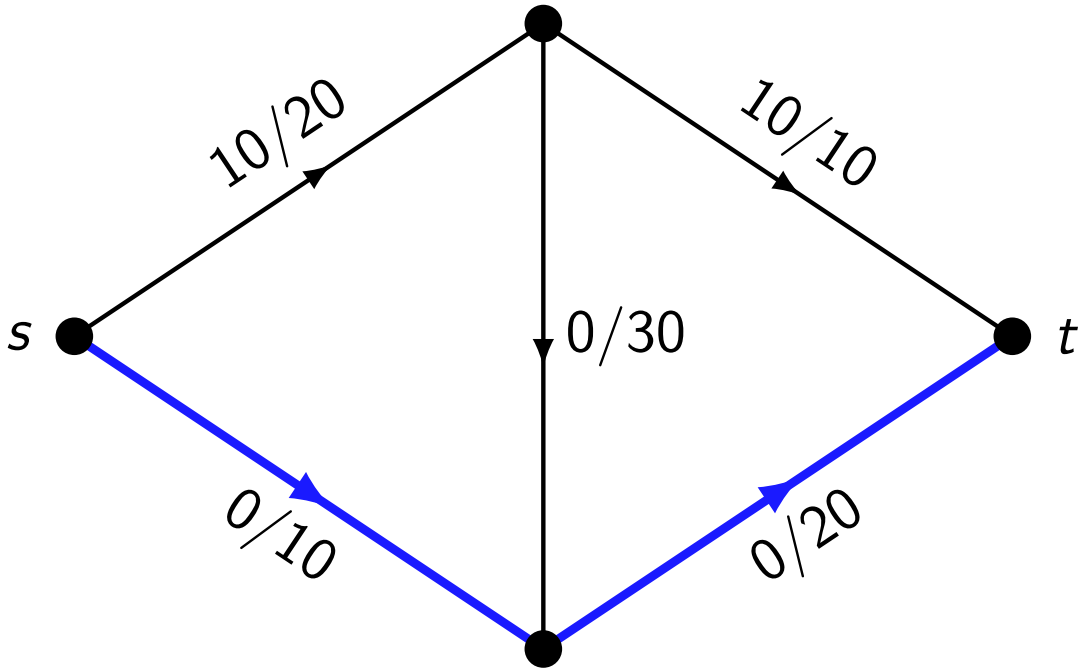

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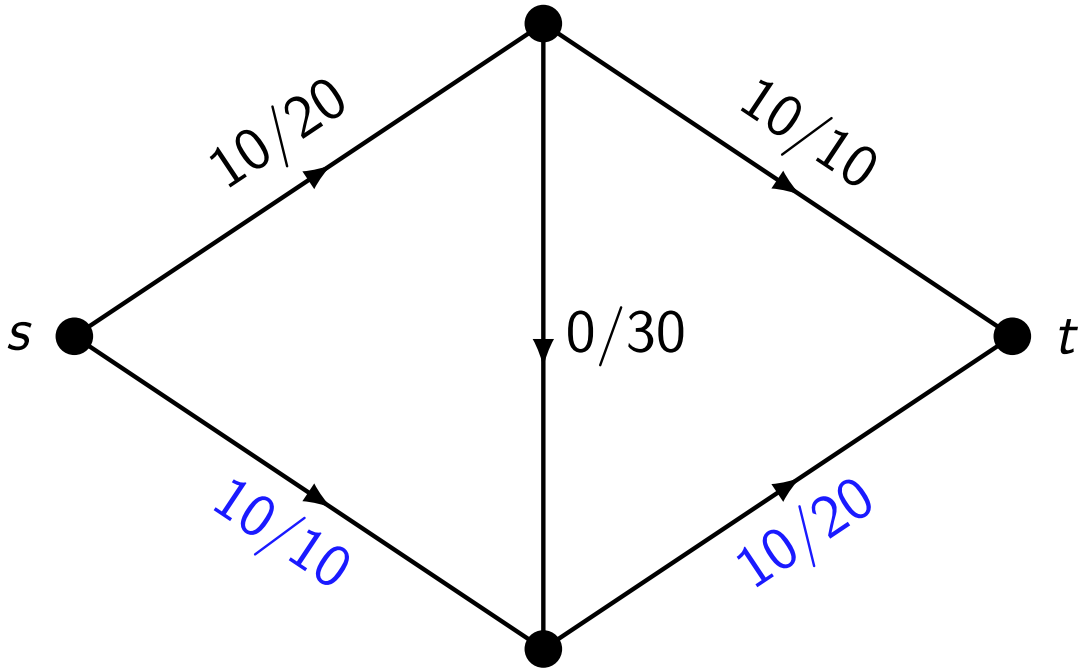

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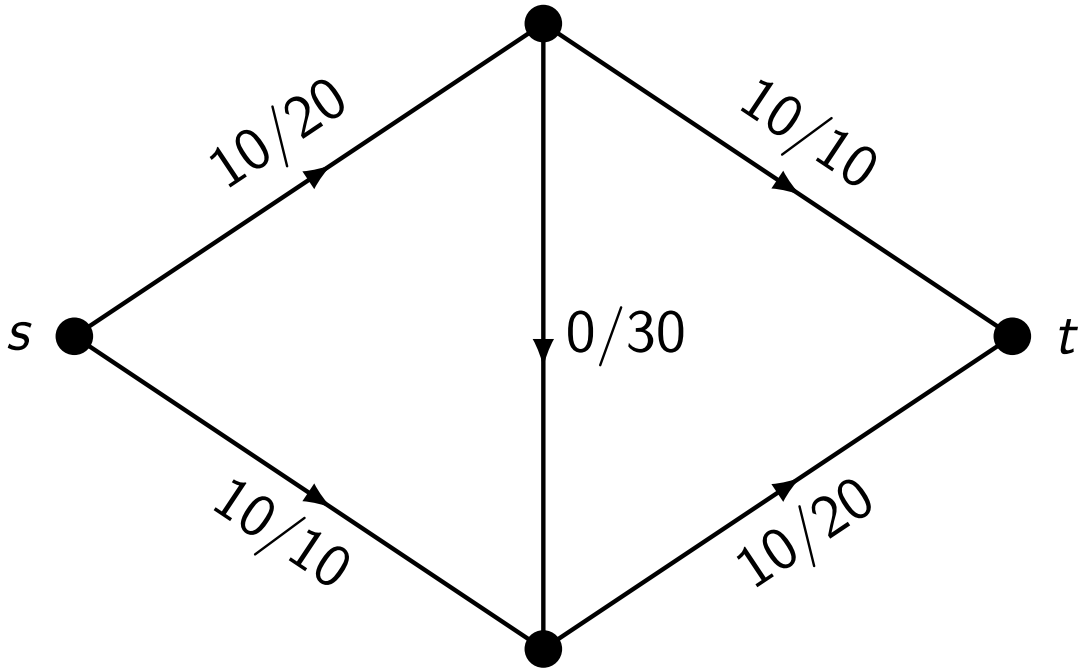

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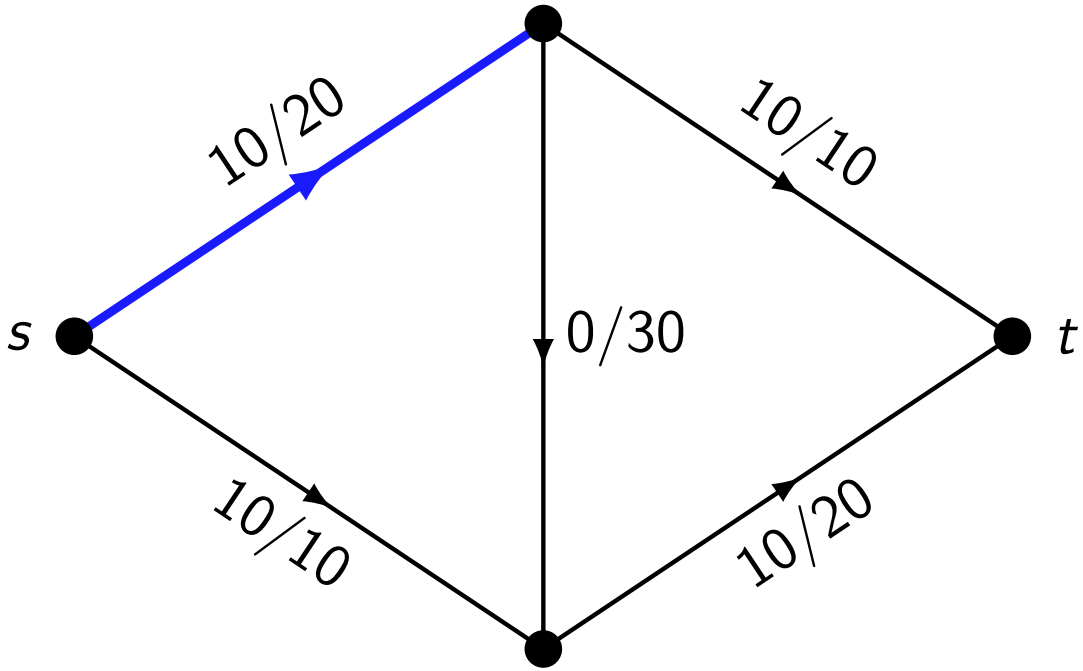

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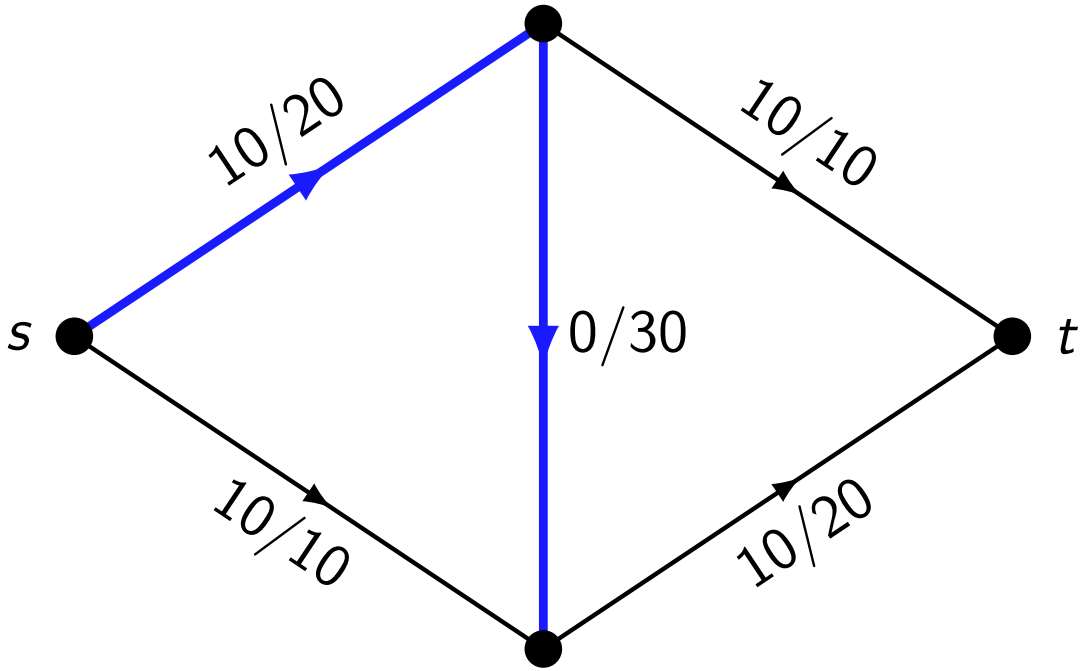

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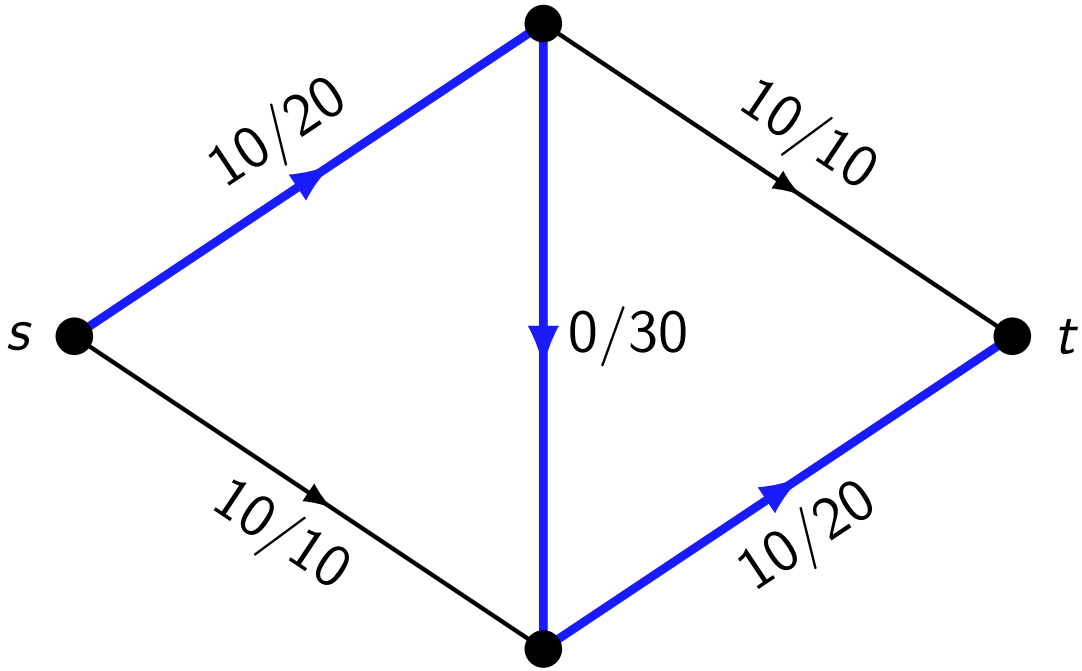

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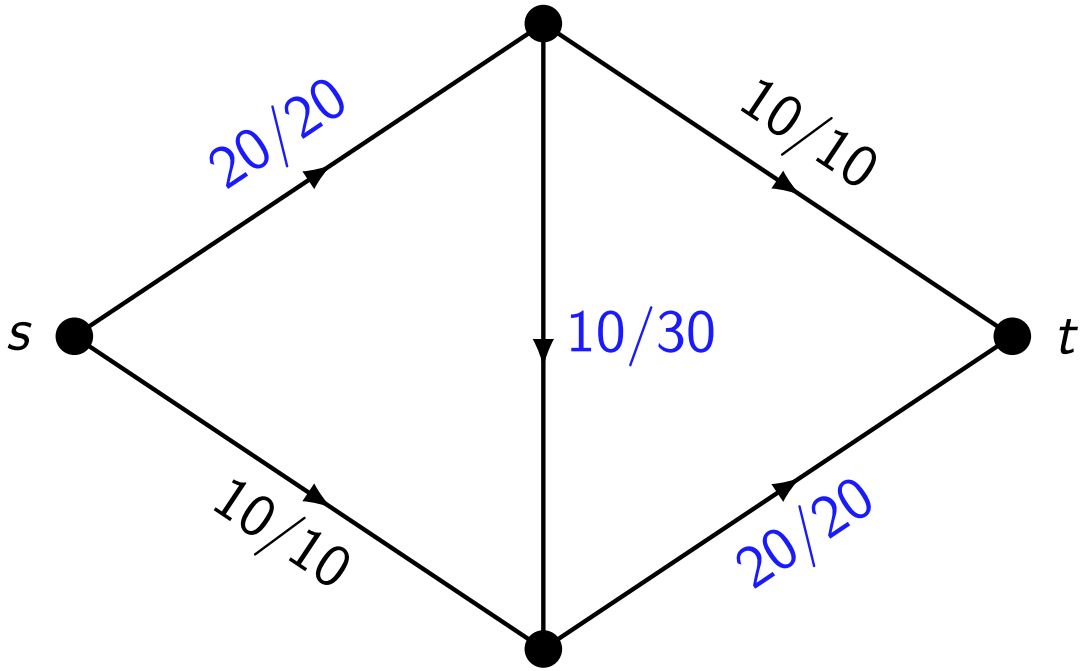

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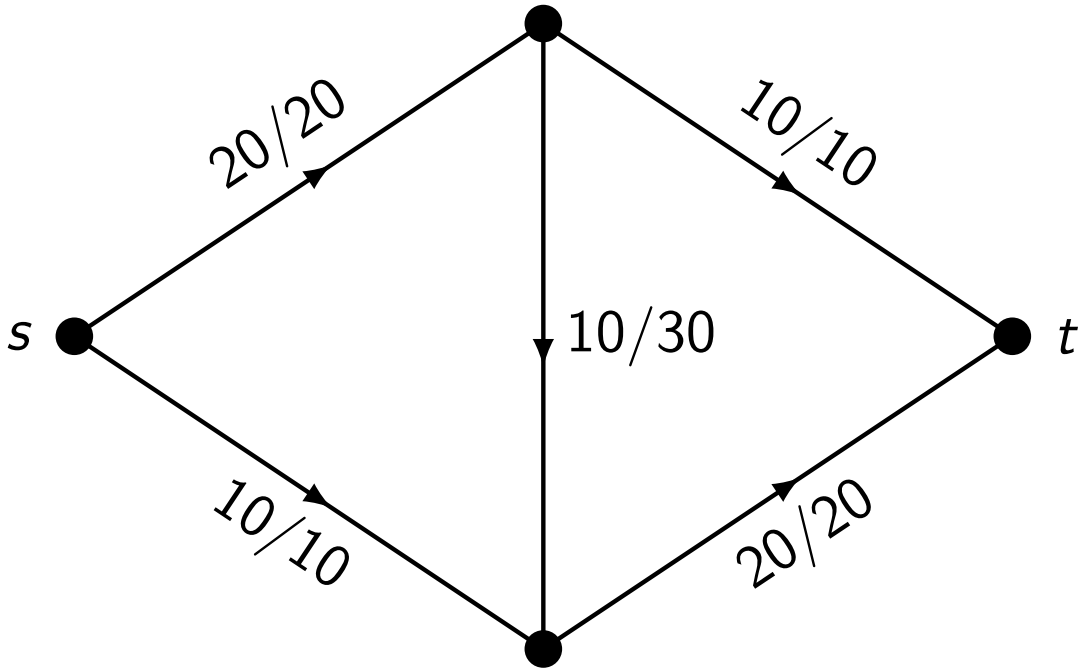

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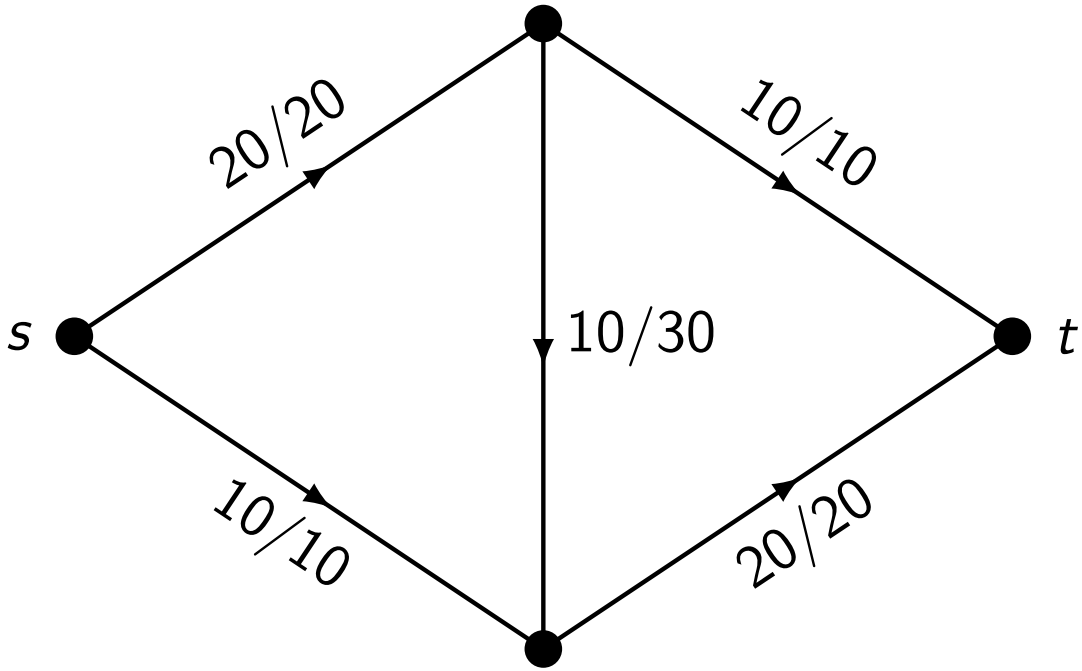

This flow has value $20 + 10 = 30$, which is best possible.
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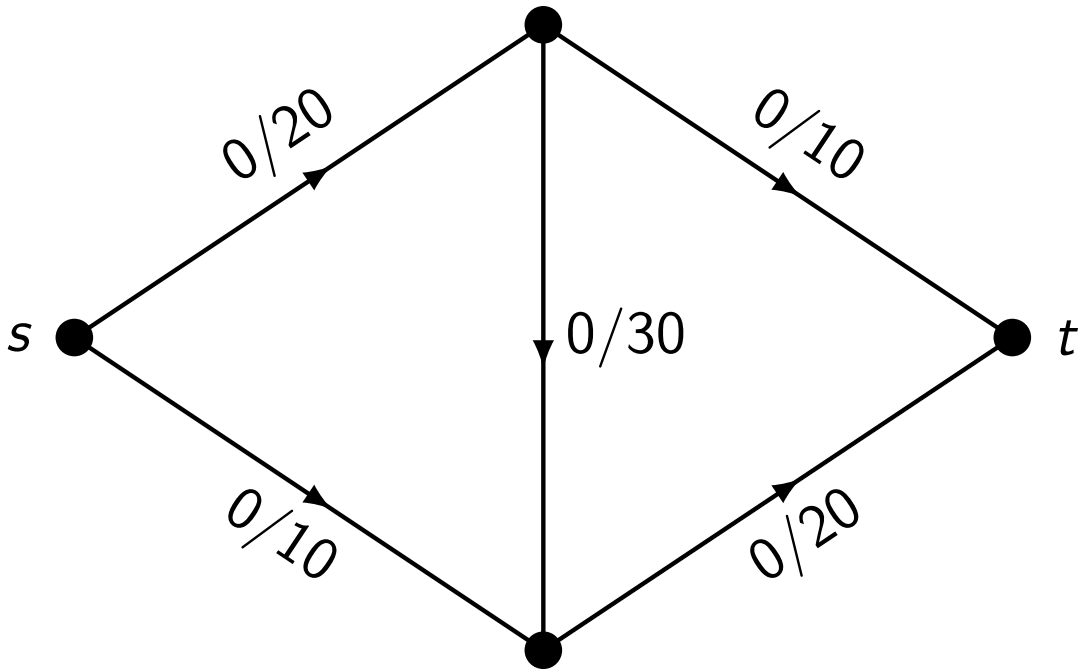

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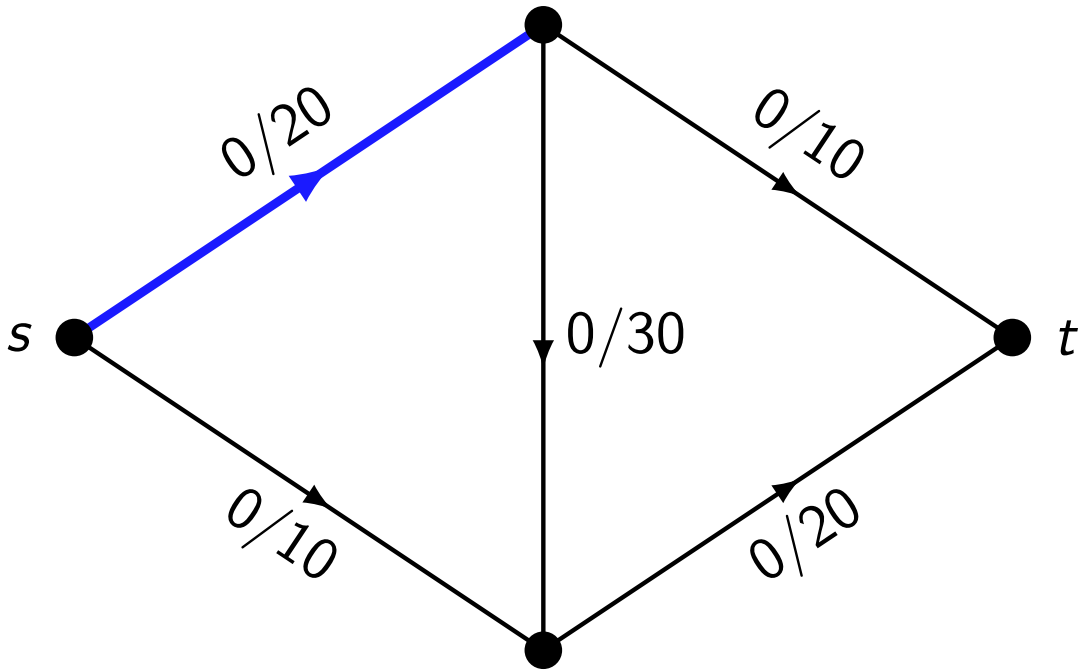

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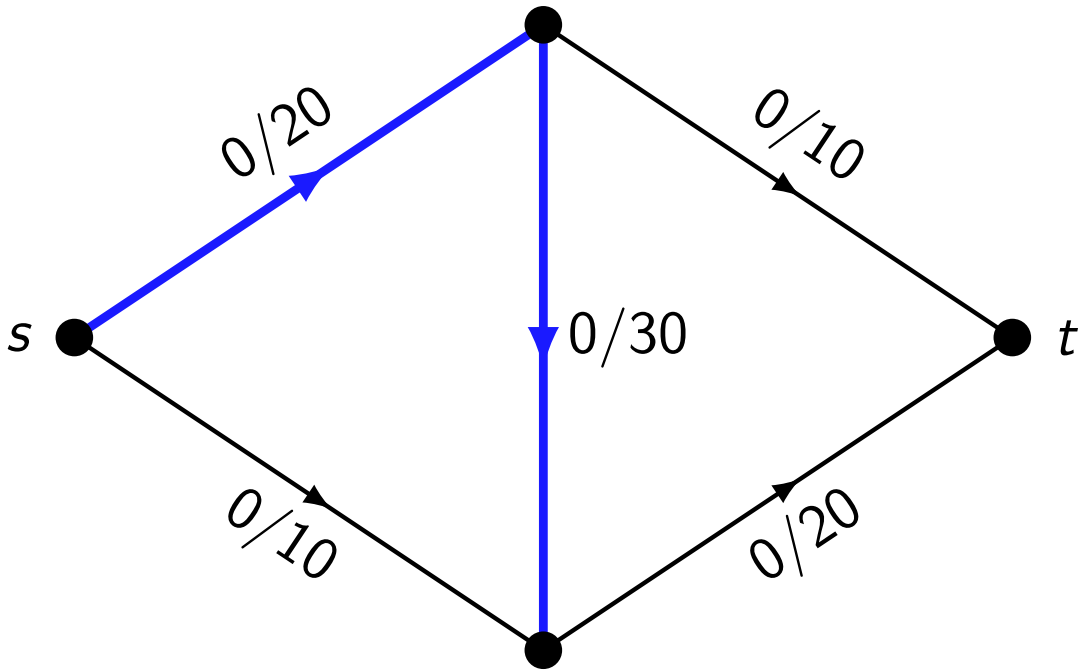

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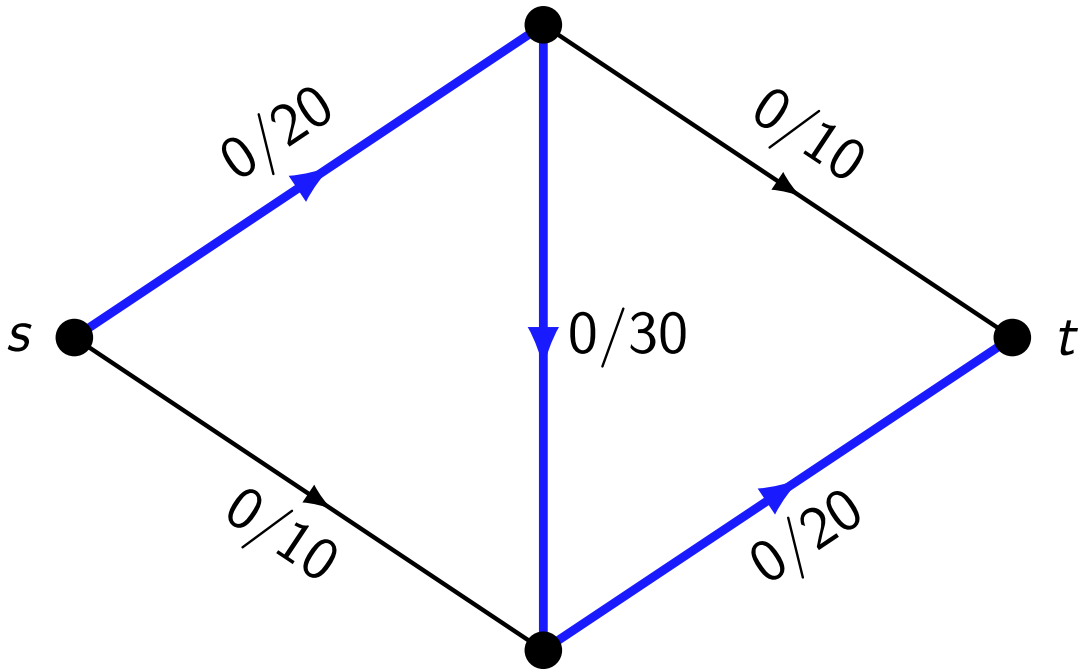

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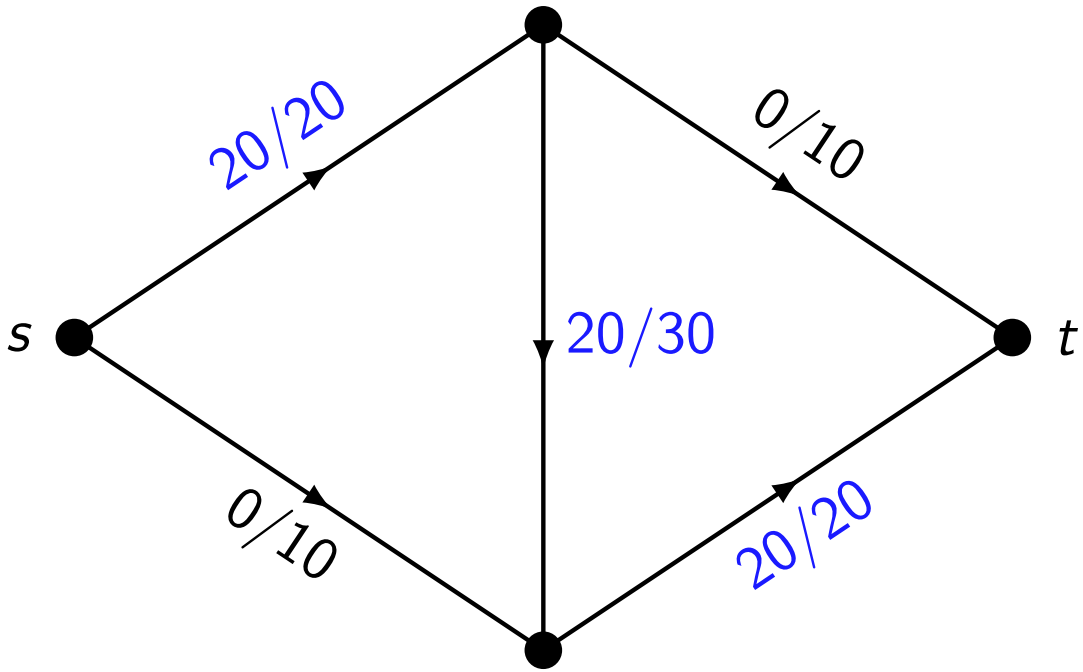

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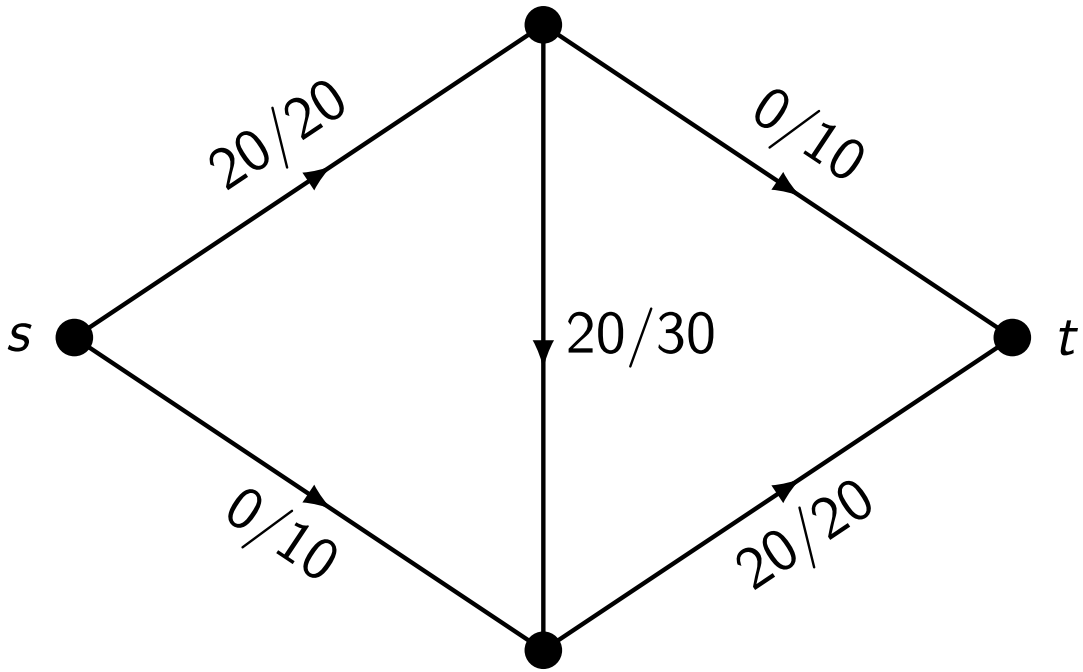

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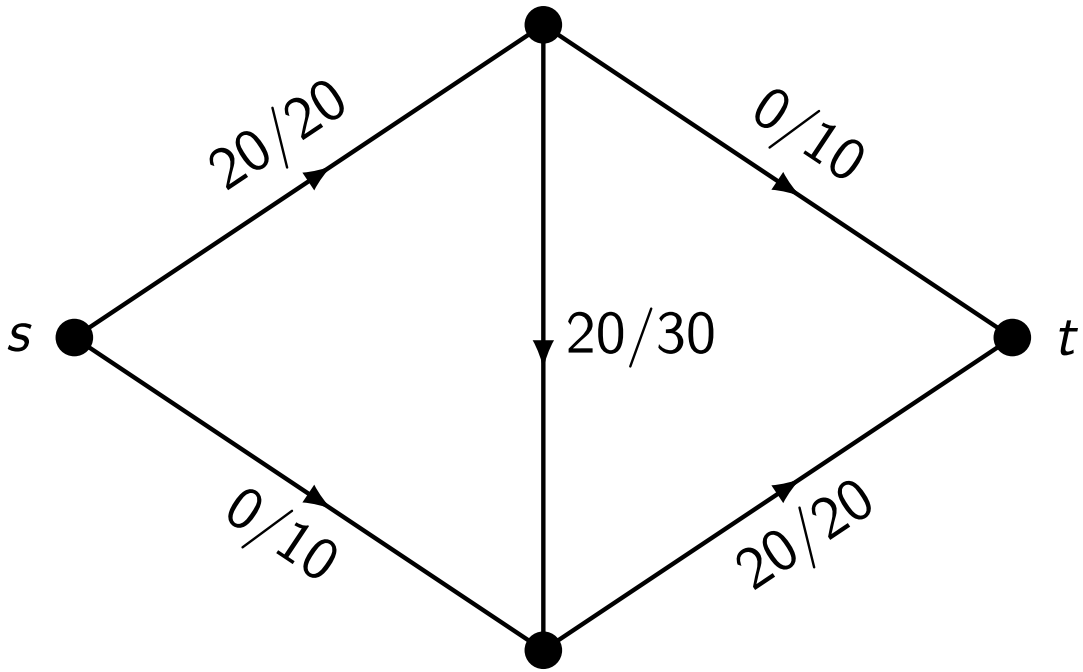

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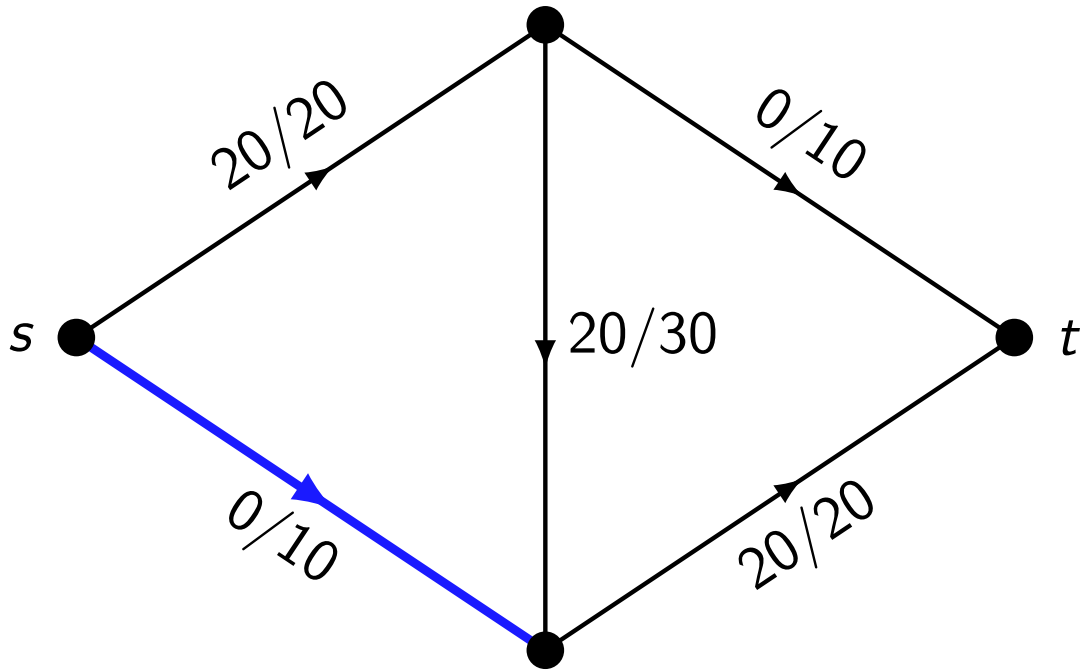

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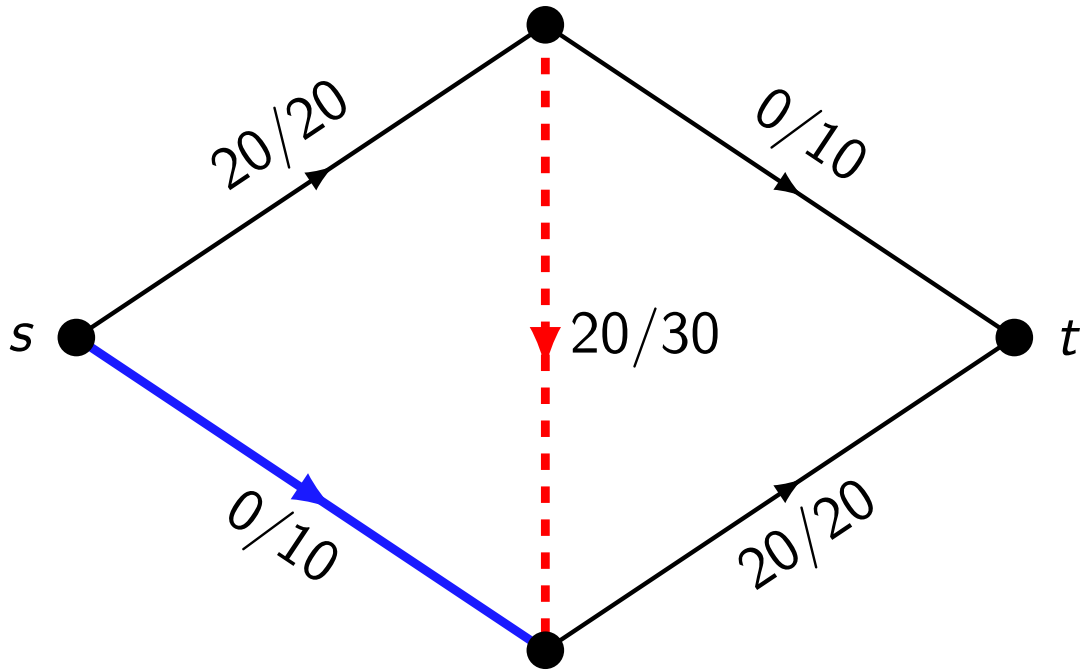

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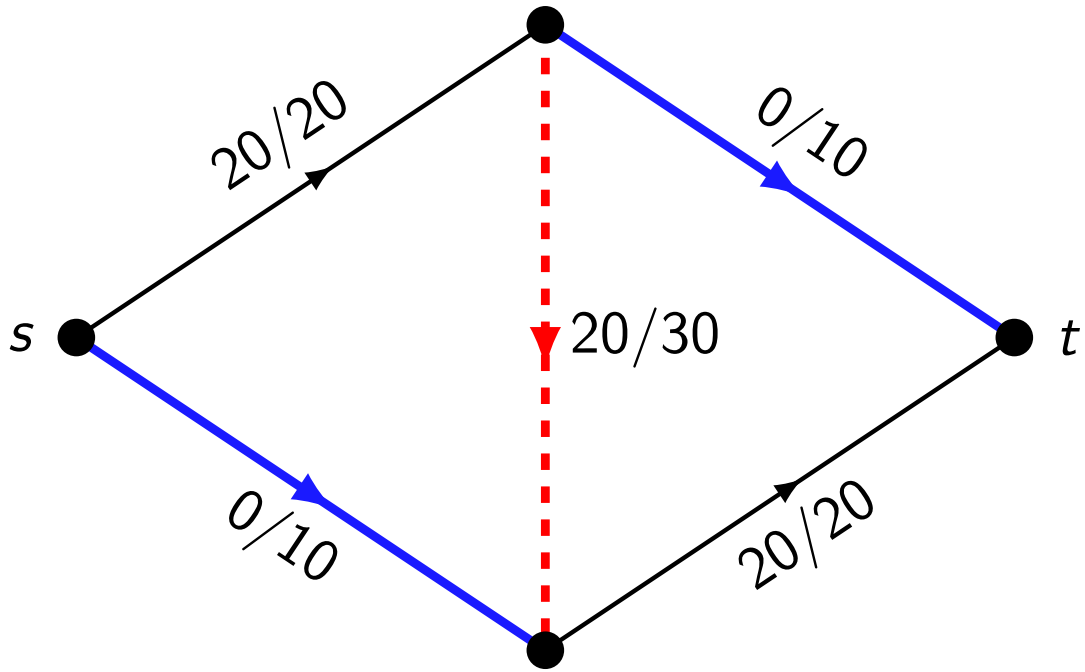

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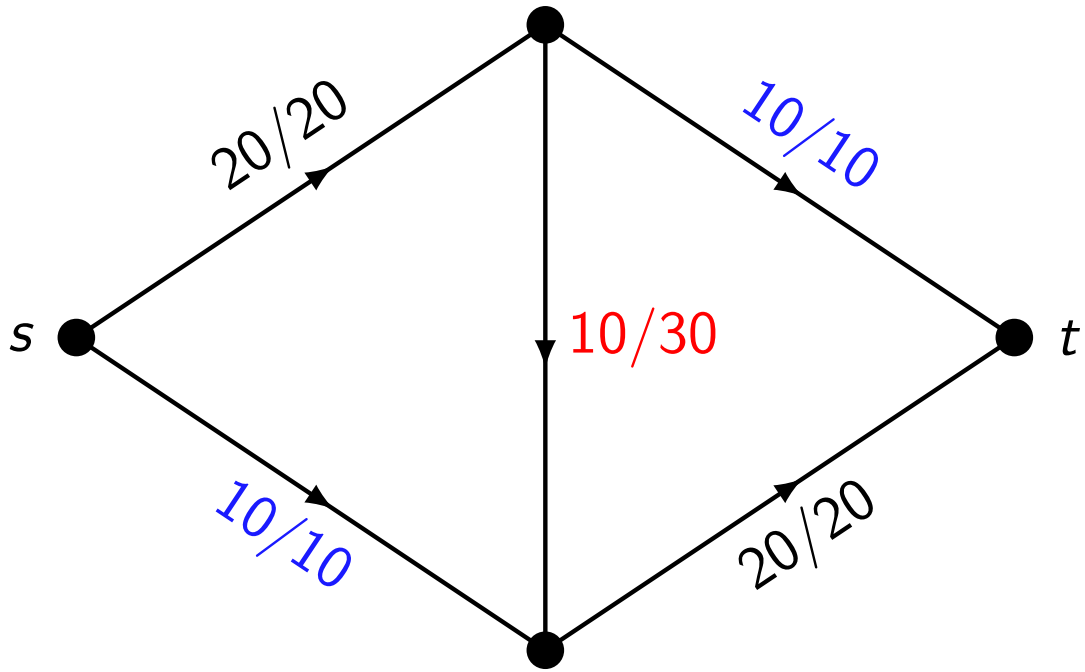

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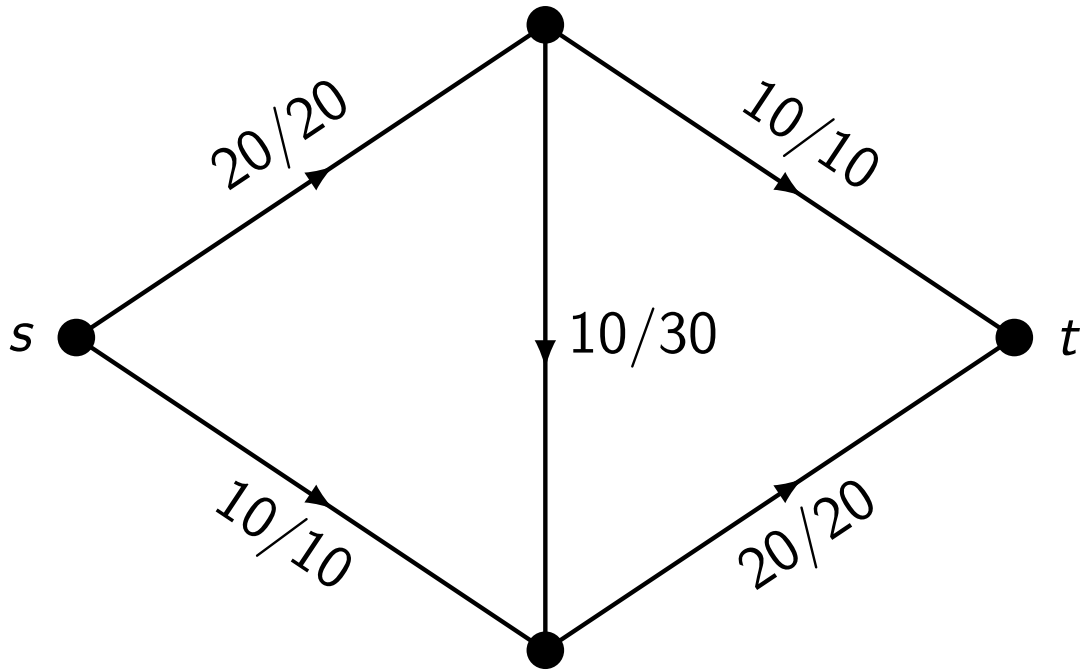

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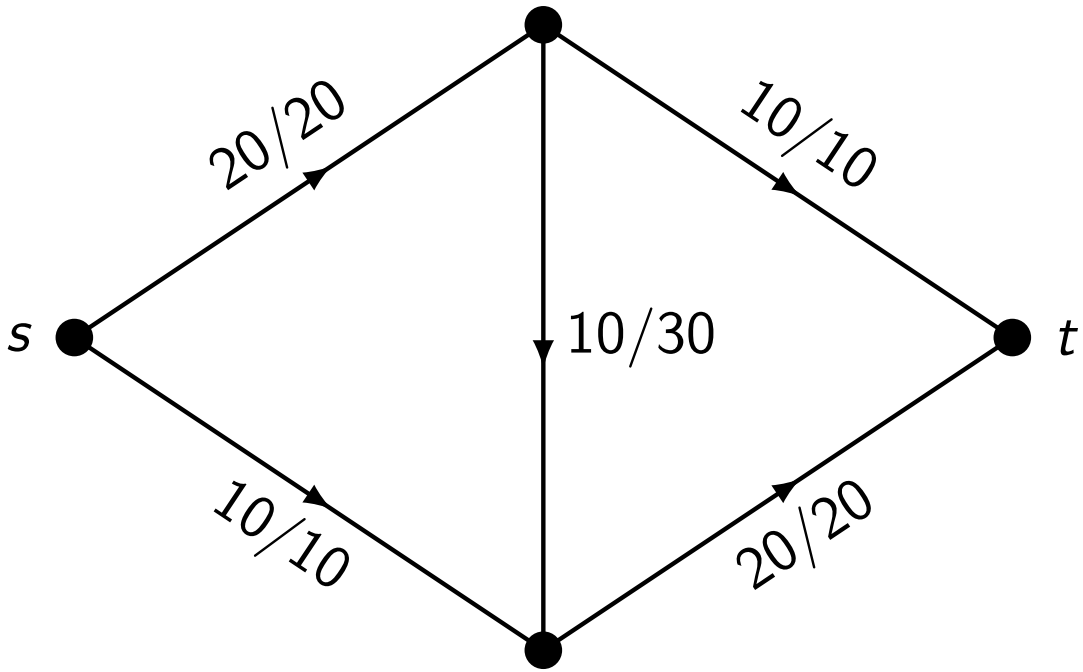

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We get a maximum flow! Now to generalise this...

A **flow** is a function $f : E \to \mathbb{R}$ such that for all $e \in E$ and $v \in V \setminus \{s, t\}$:

- $0 \leq f(e) \leq c(e)$;
- $f^+(v) := \sum_{w \in N^+(v)} f(v, w) = \sum_{u \in N^-(v)} f(u, v) =: f^-(v)$.

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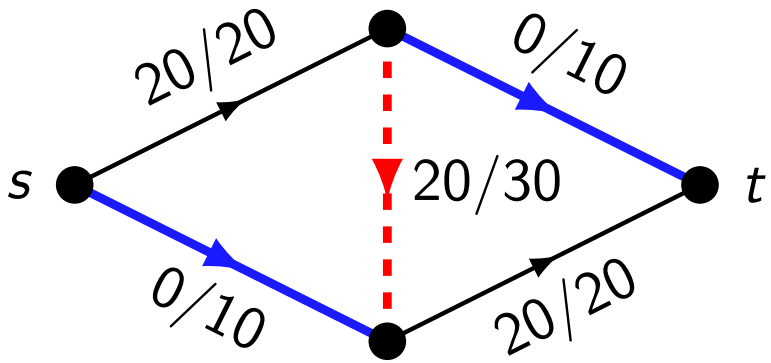

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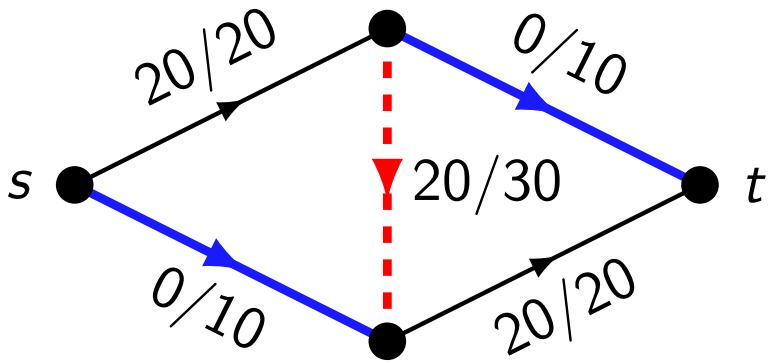

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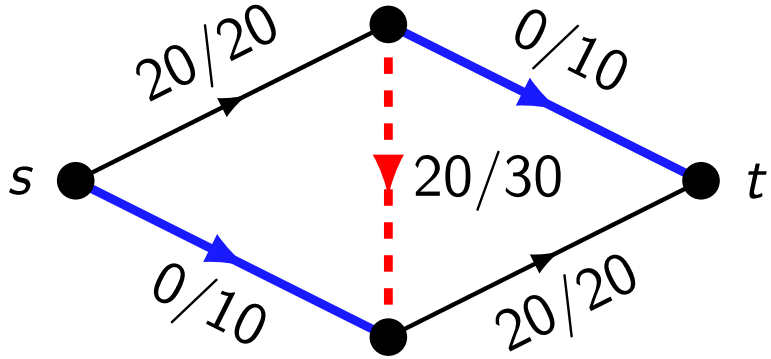
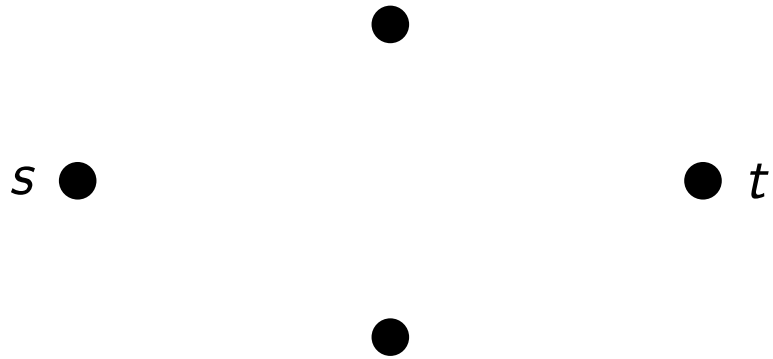
But there's an annoying technicality with bidirected edges...





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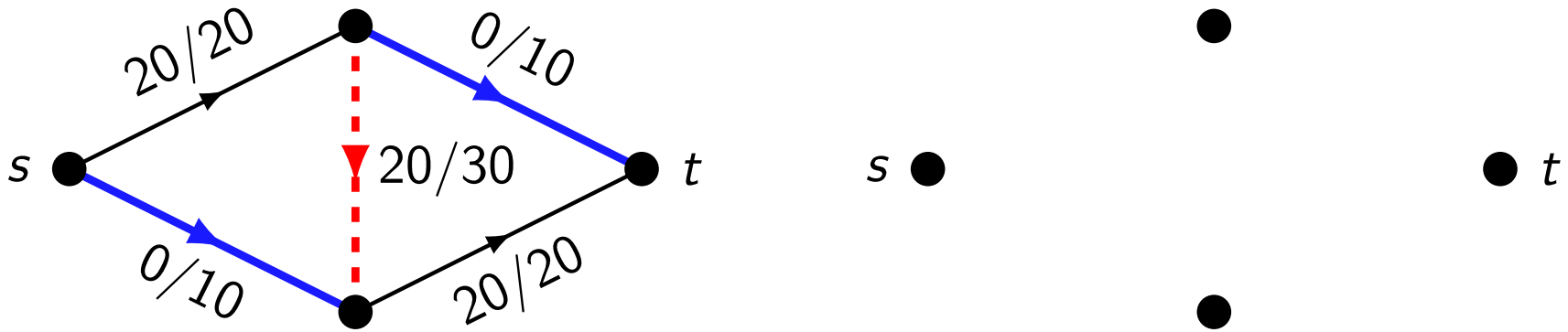

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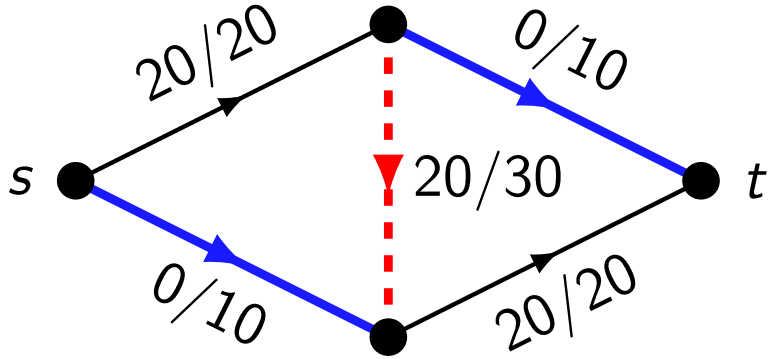

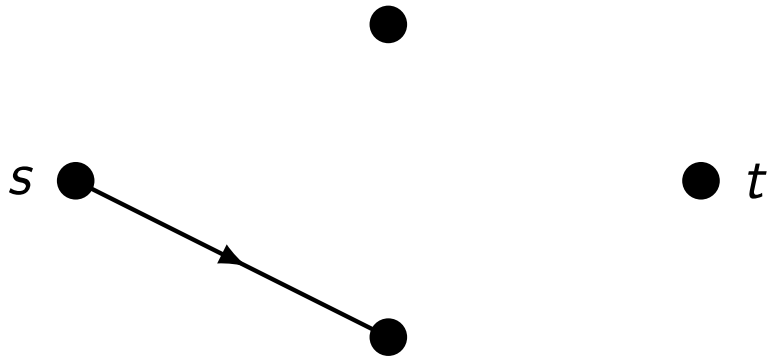

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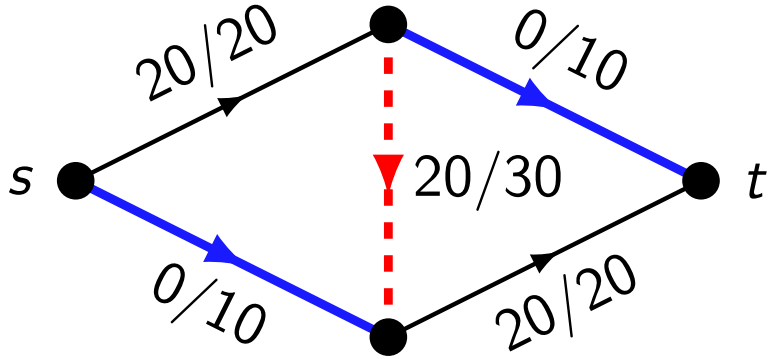

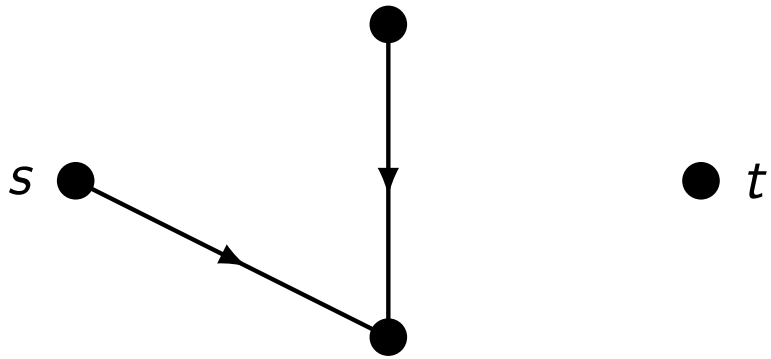

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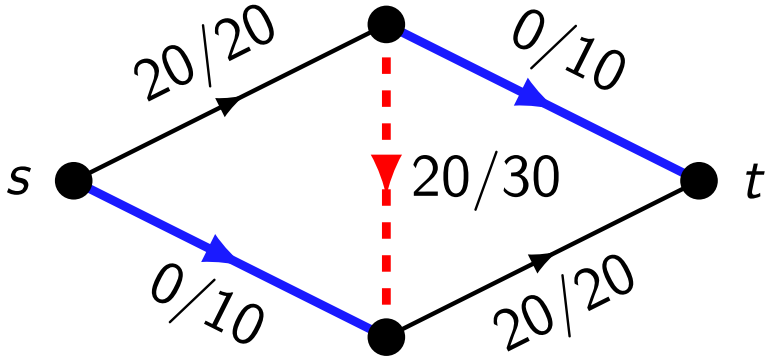

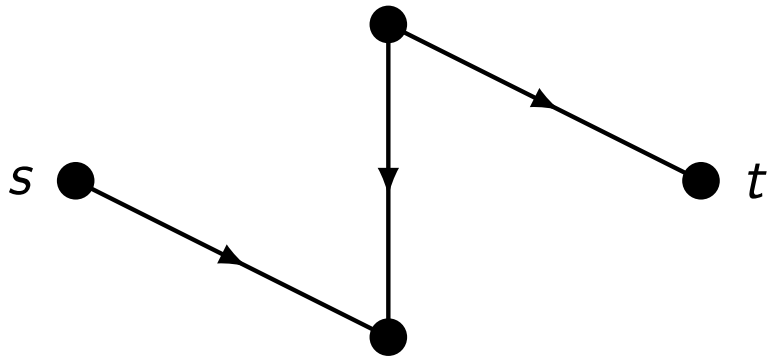

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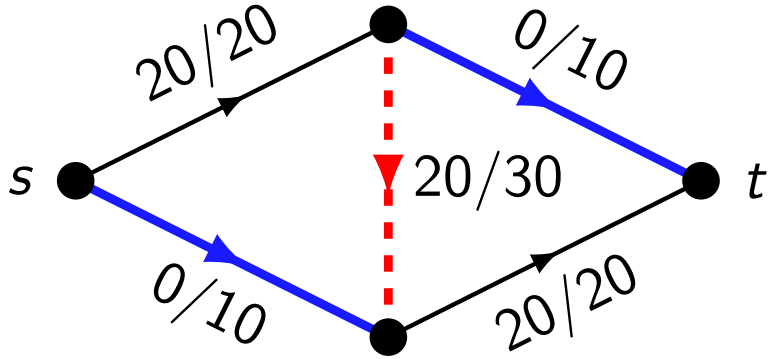

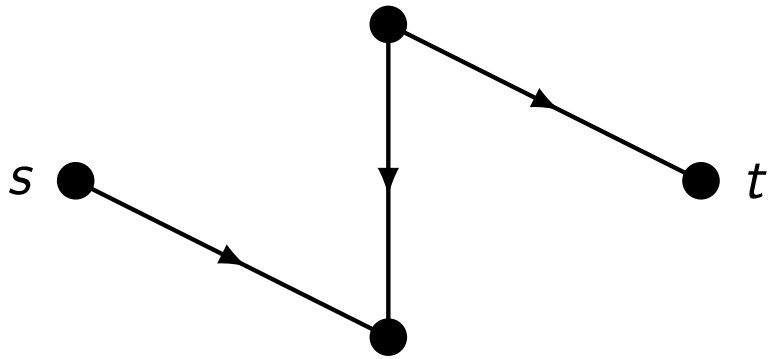

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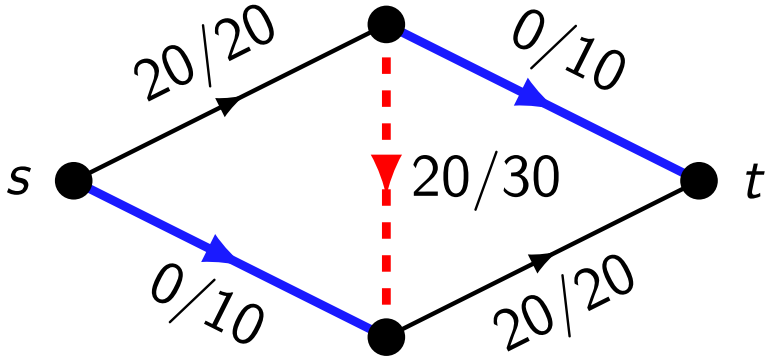

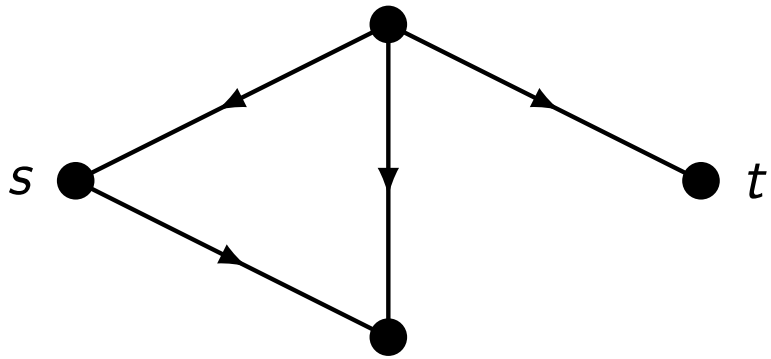

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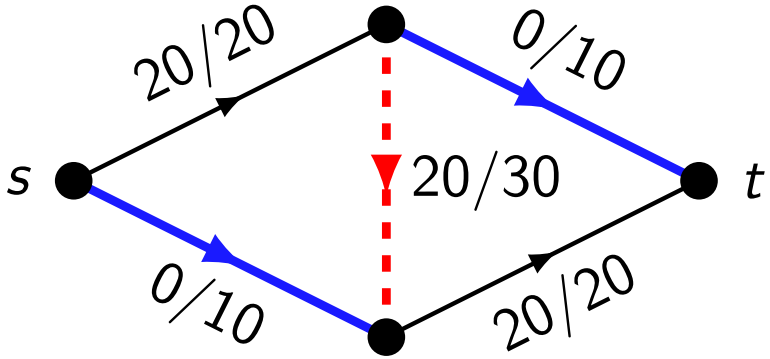

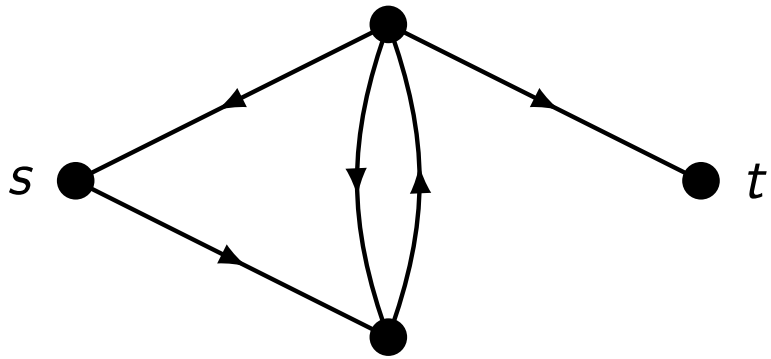

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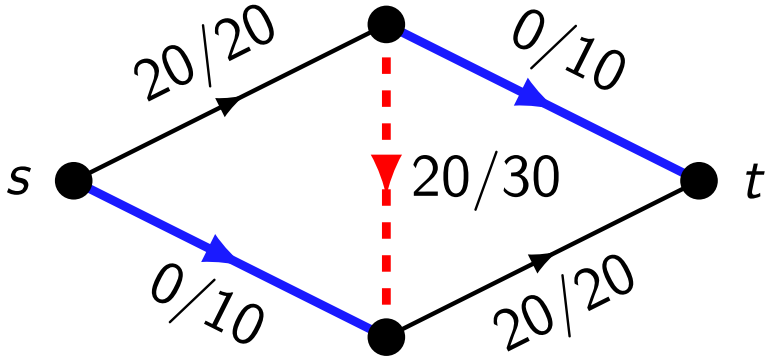

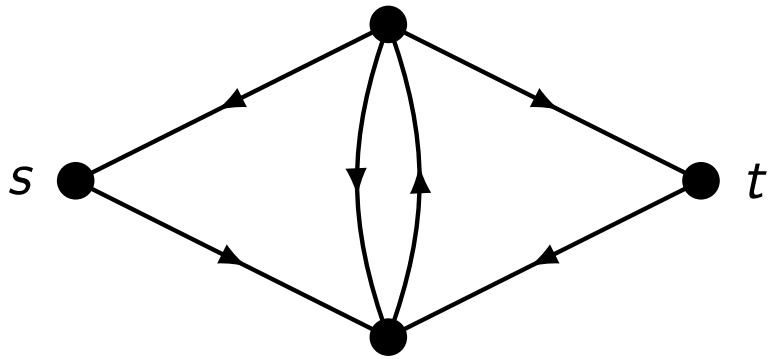

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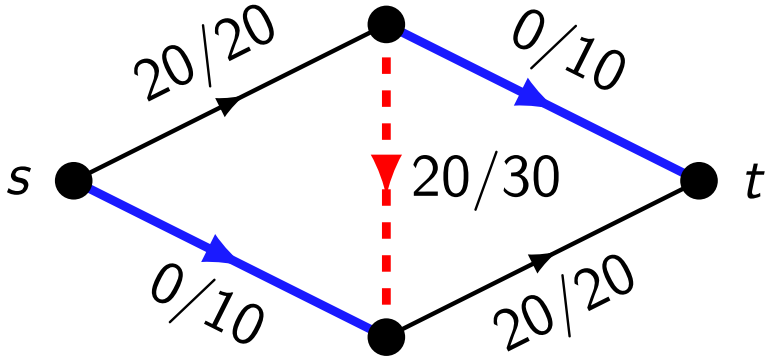

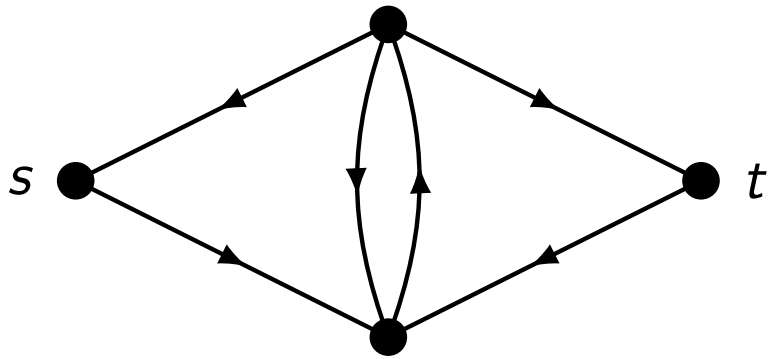

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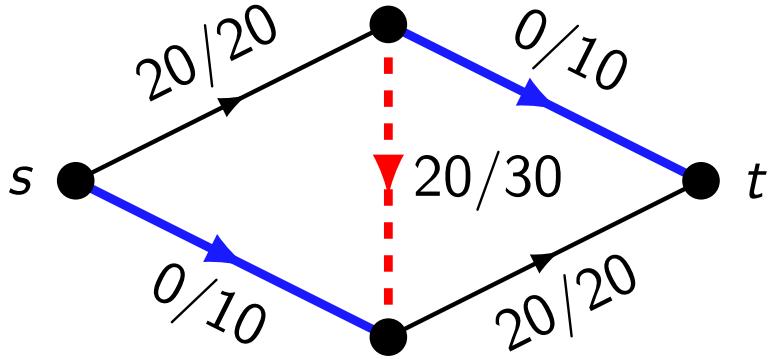

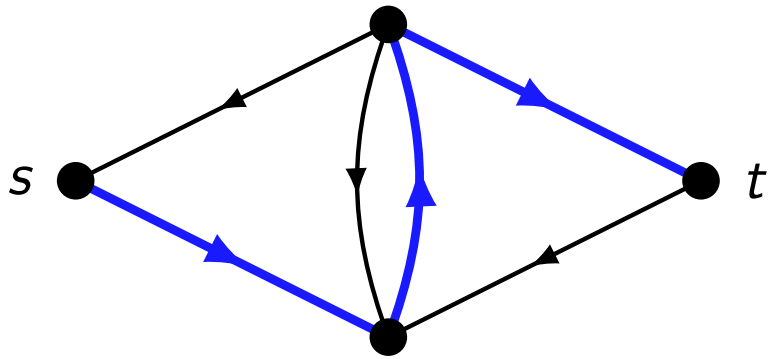

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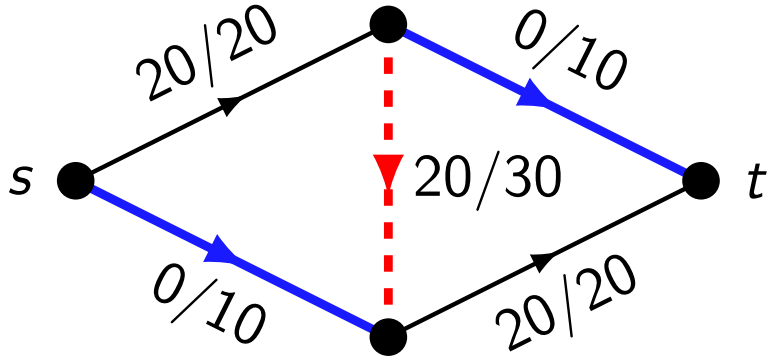

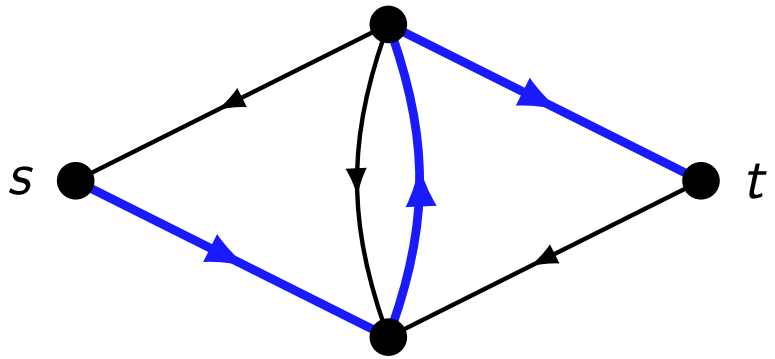

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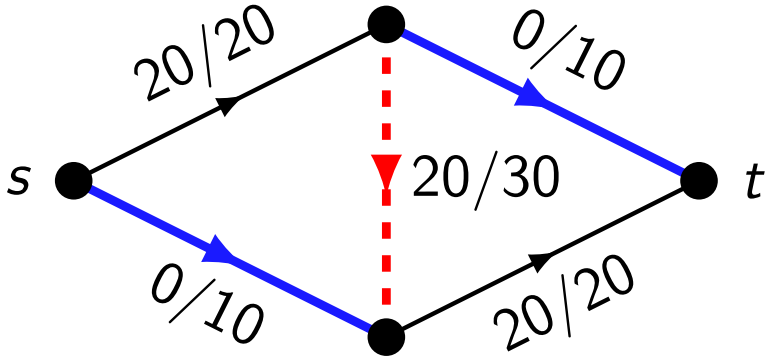

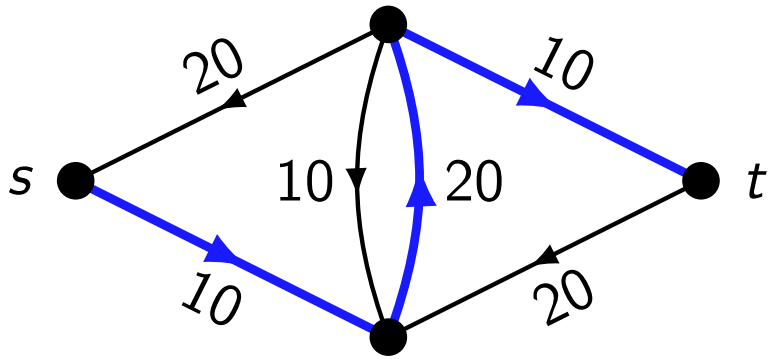

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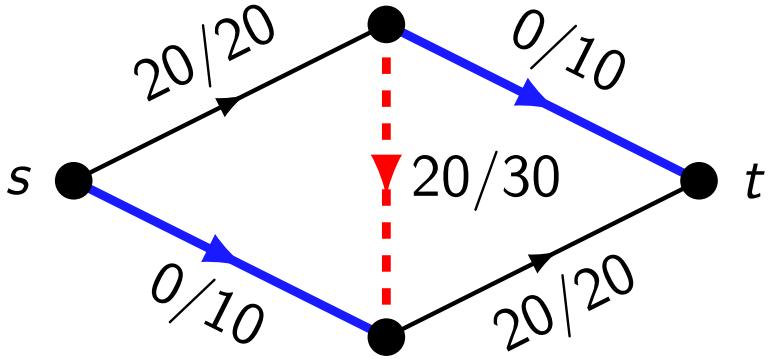

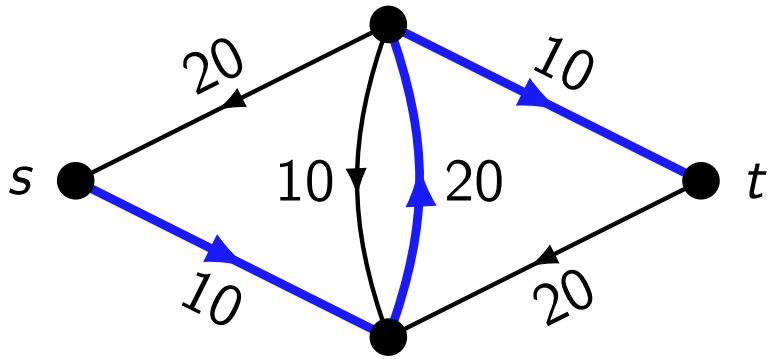

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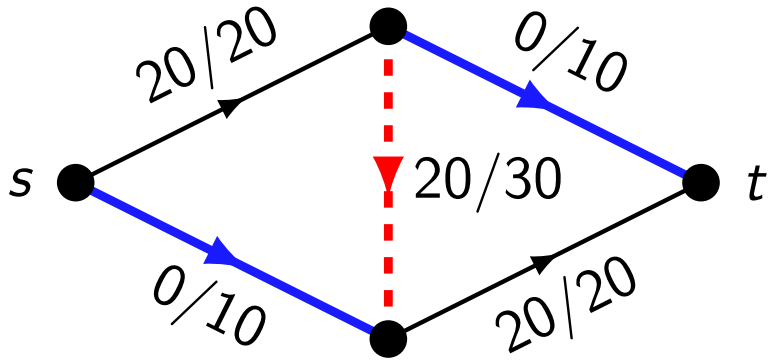

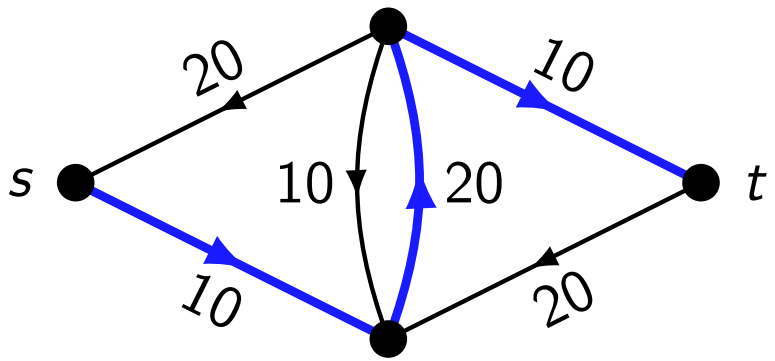

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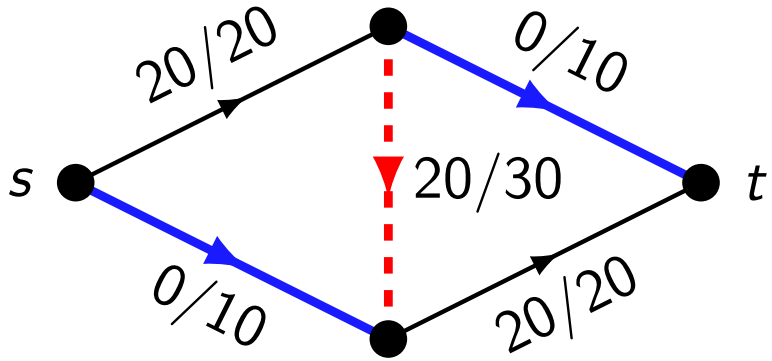

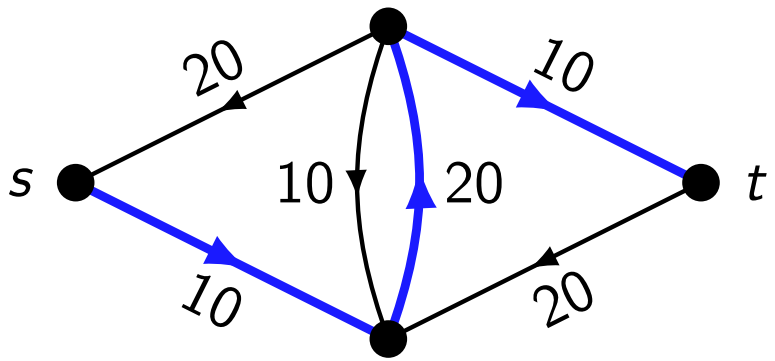

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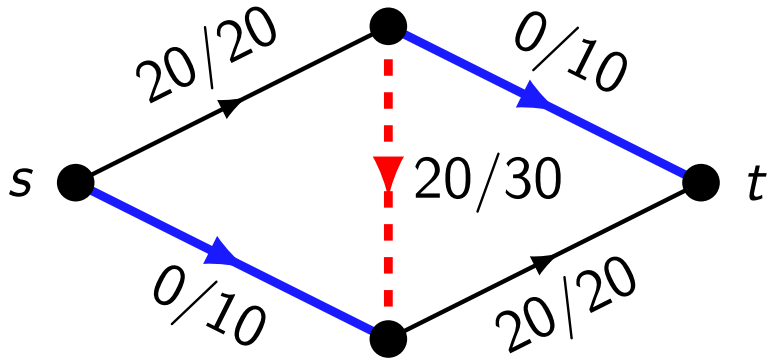

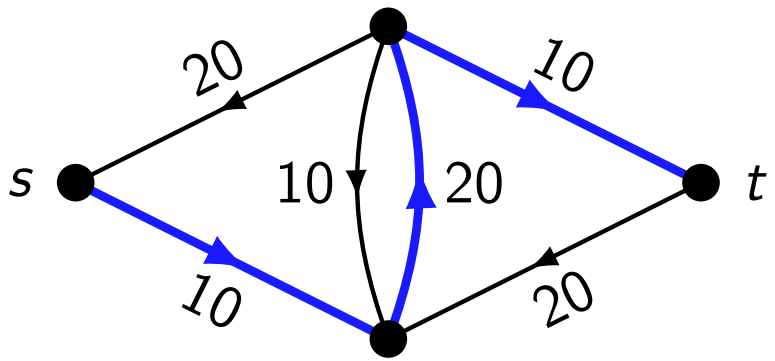

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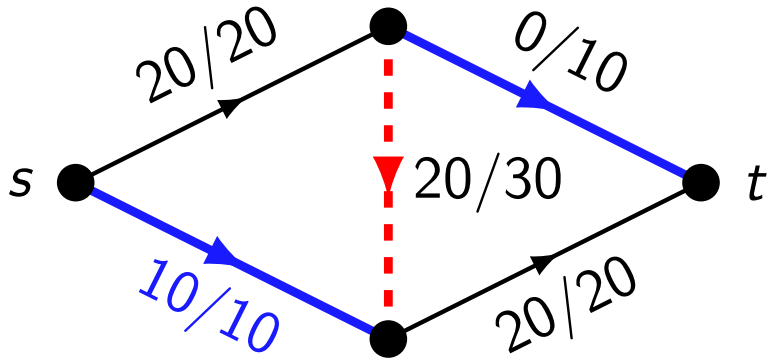

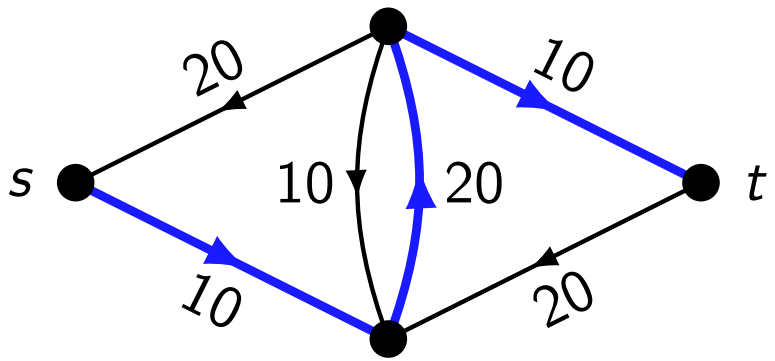

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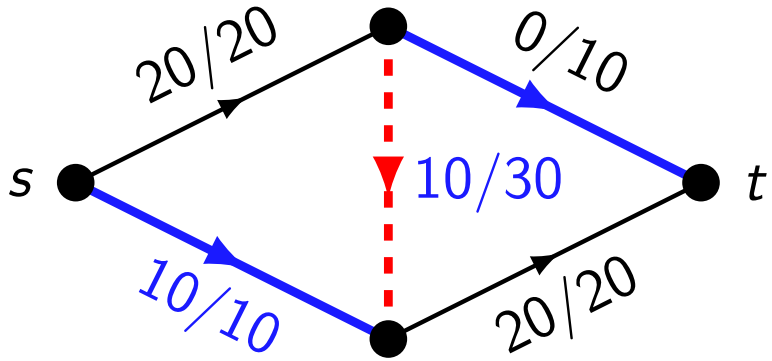

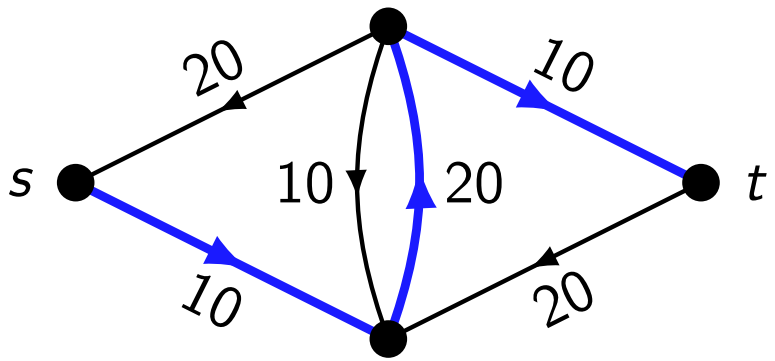

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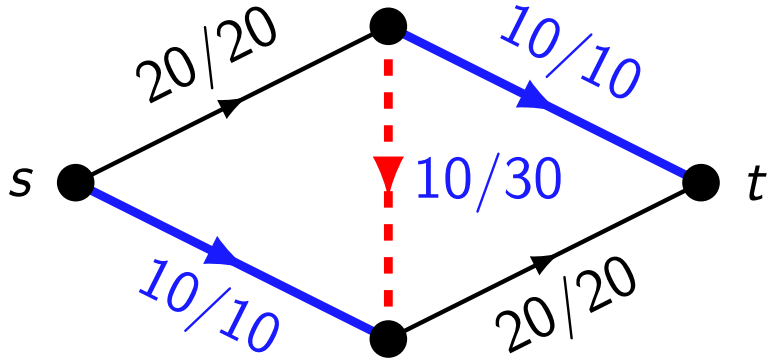

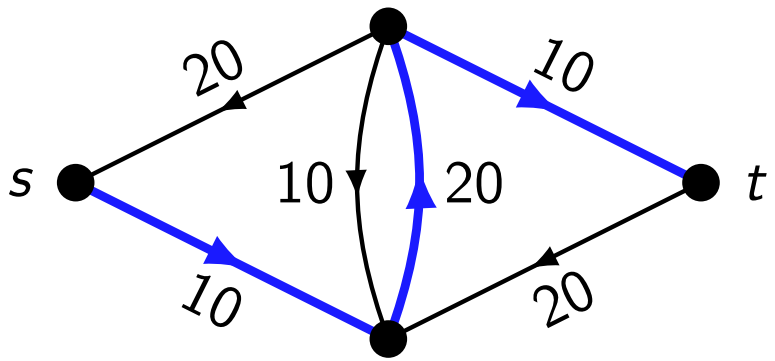

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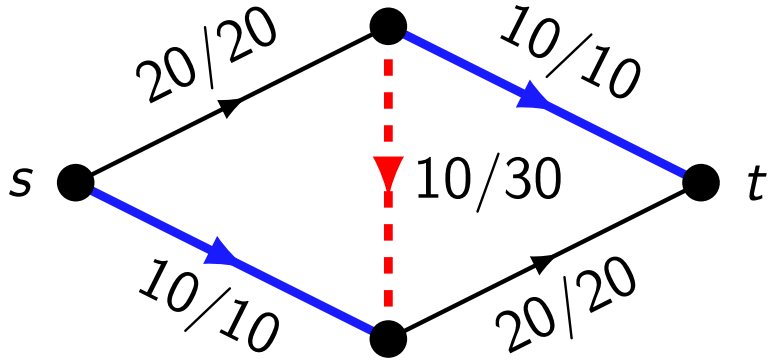

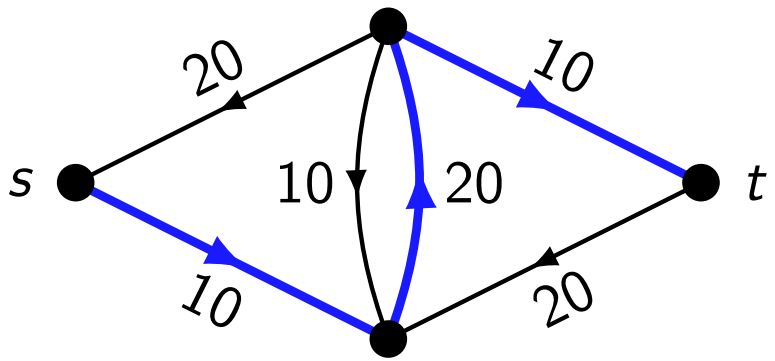

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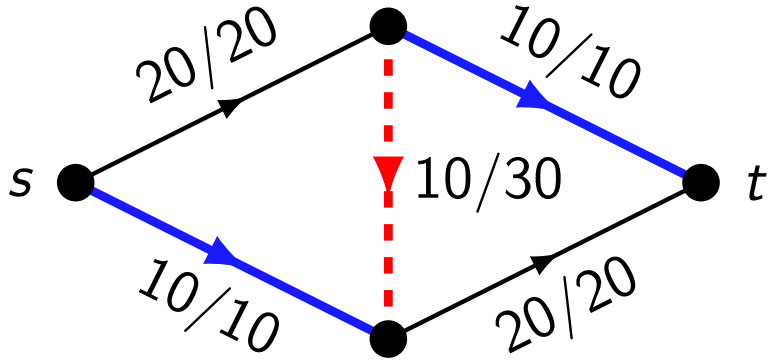

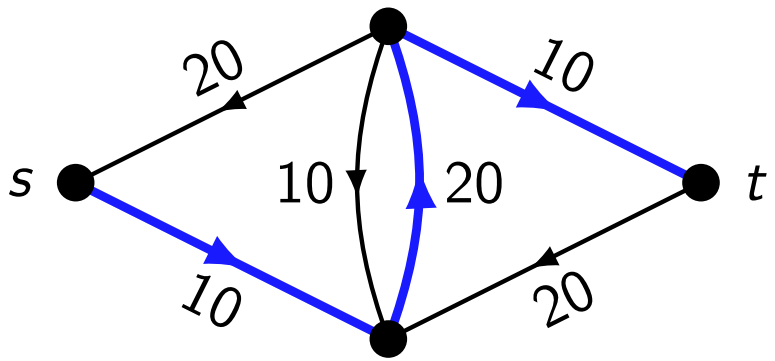

Define $\texttt{Push}(G, c, s, t, f, P)$ as follows:

- Let C be the residual capacity of P. (Here C is 10.)
- For each edge (u, v) of P: if $c(u, v) - f(u, v) \geq C$, then add C to $f(u, v)$; otherwise, we have $f(v, u) \geq C$, so subtract C from $f(v, u)$.

Lemma 3: $\texttt{Push}(G, c, s, t, f, P)$ returns a new flow f', with value $v(f') = v(f) + C$, in $O(|V(G)|)$ time. □

The Ford-Fulkerson Algorithm

Algorithm: FordFulkerson

Input : A (weakly connected) flow network (G, c, s, t).
Output : A flow f with no augmenting paths.

1 **begin**
2 Construct the flow f with $f(e) = 0$ for all $e \in E(G)$.
3 Construct the residual graph G_f.
4 **while** *G_f contains a path P from s to t* **do**
5 Find P using depth-first (or breadth-first) search.
6 Update $f \leftarrow \texttt{Push}(G, c, s, t, f, P)$.
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By Lemma 3, every iteration of 4–7 increases $v(f)$ by at least 1. So if f^* is a maximum flow, there are at most $v(f^*)$ iterations in total.

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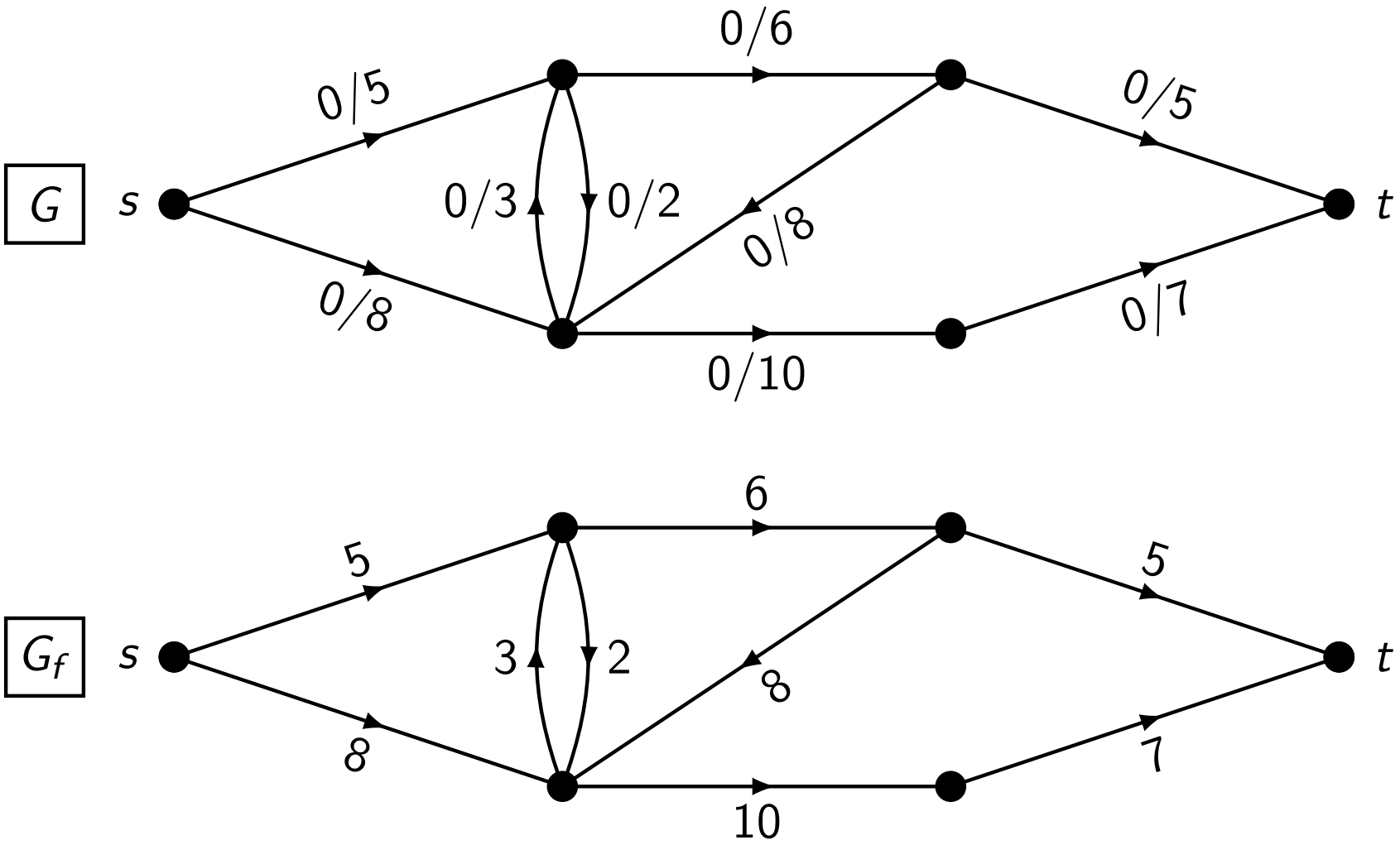
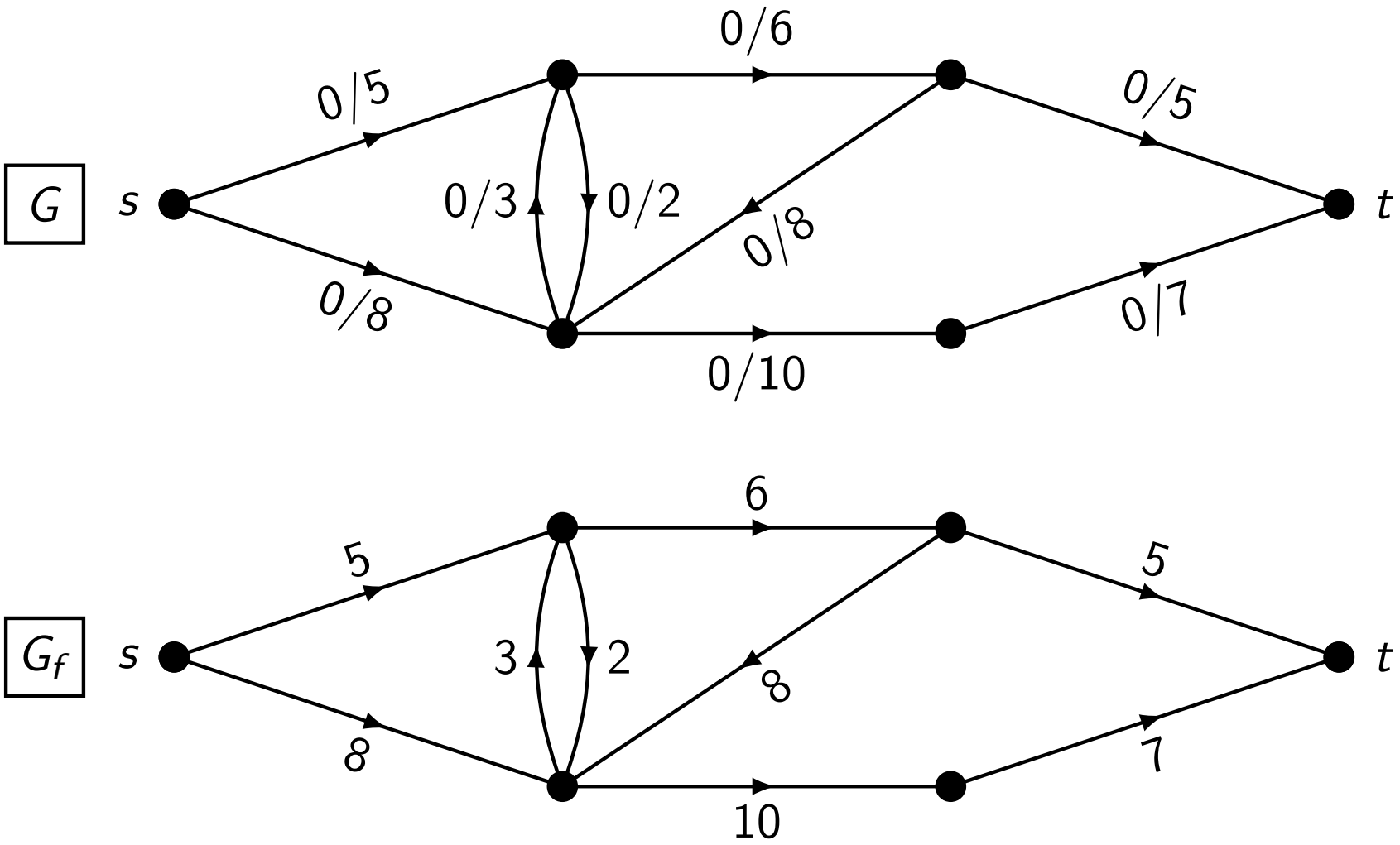
Every step takes $O(|E|)$ time or $O(|V|)$ time, and since G is weakly connected we have $|V| = O(|E|)$. So the running time is $O(v(f^*)|E|)$.

Worked example



Initialise flow and construct G_f.

Worked example



Apply depth-first search to find an augmenting path in G_f.

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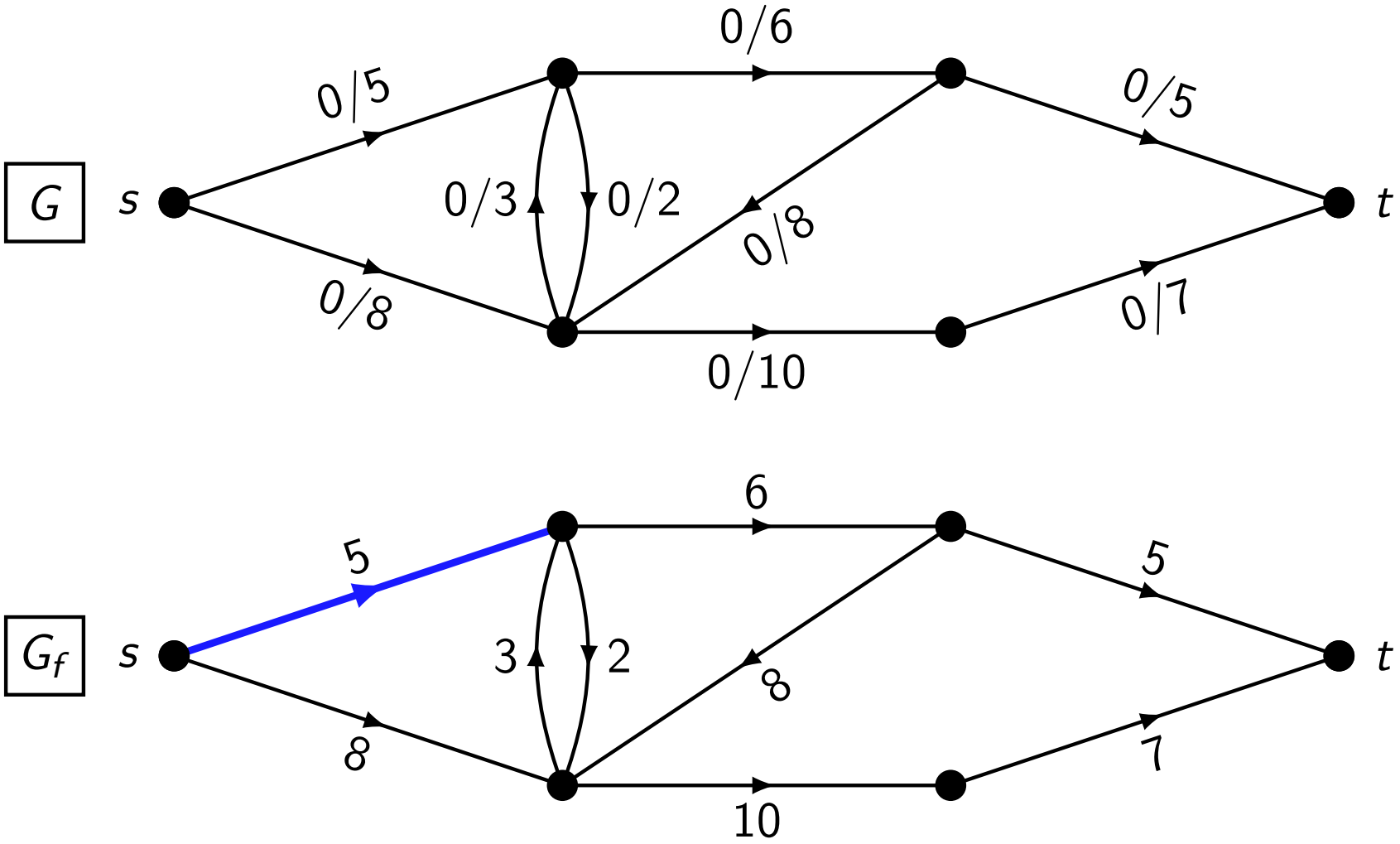

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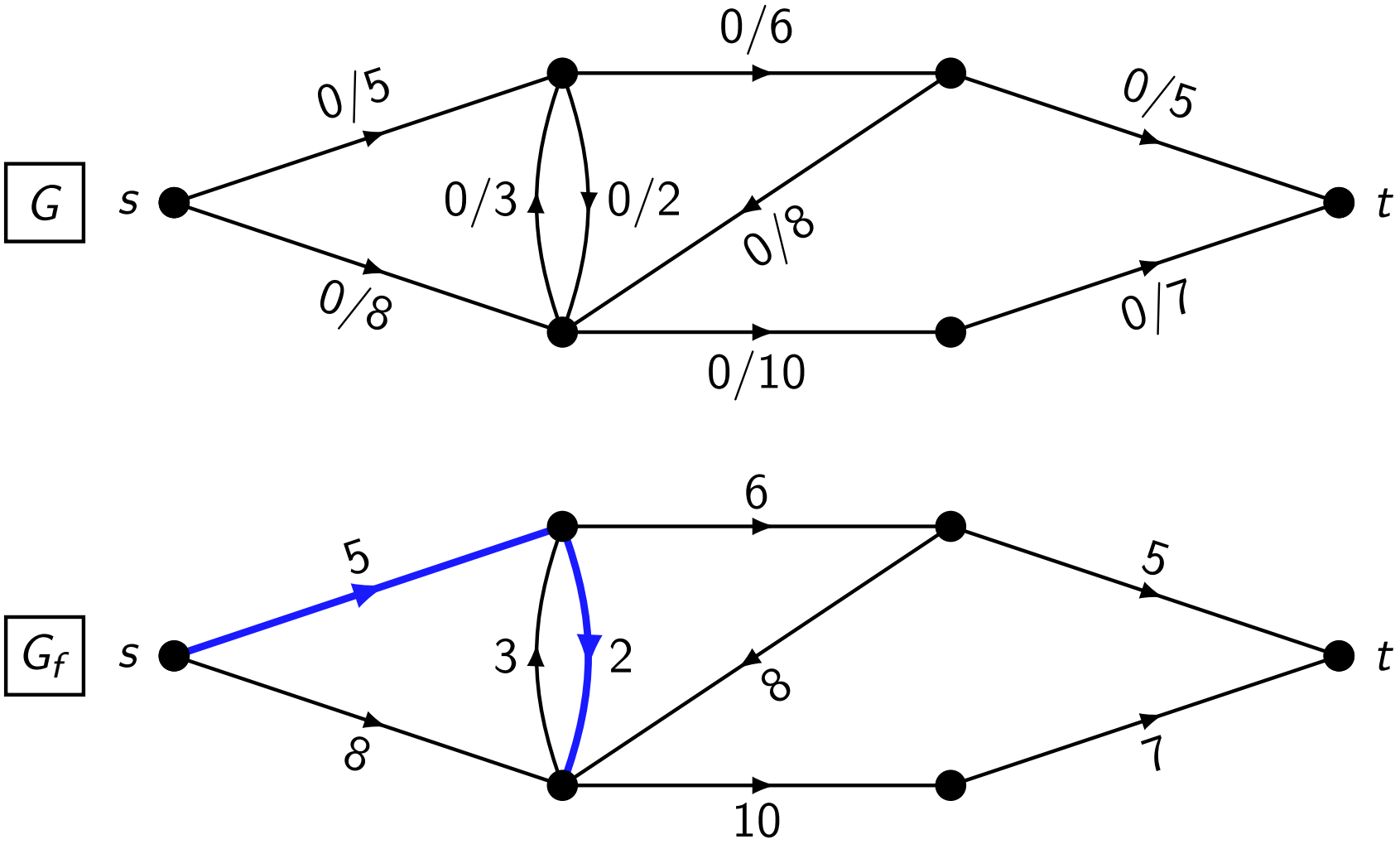

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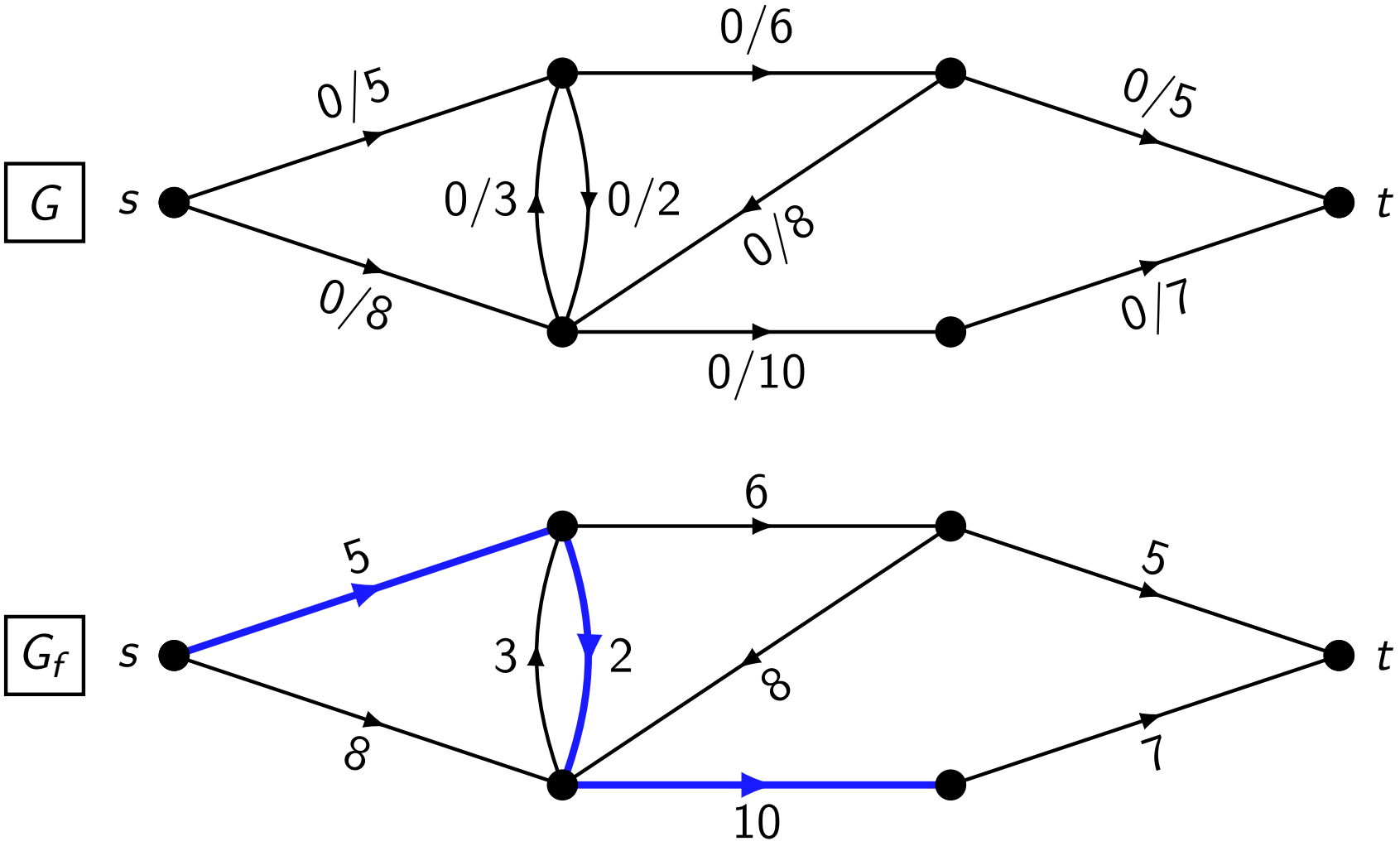

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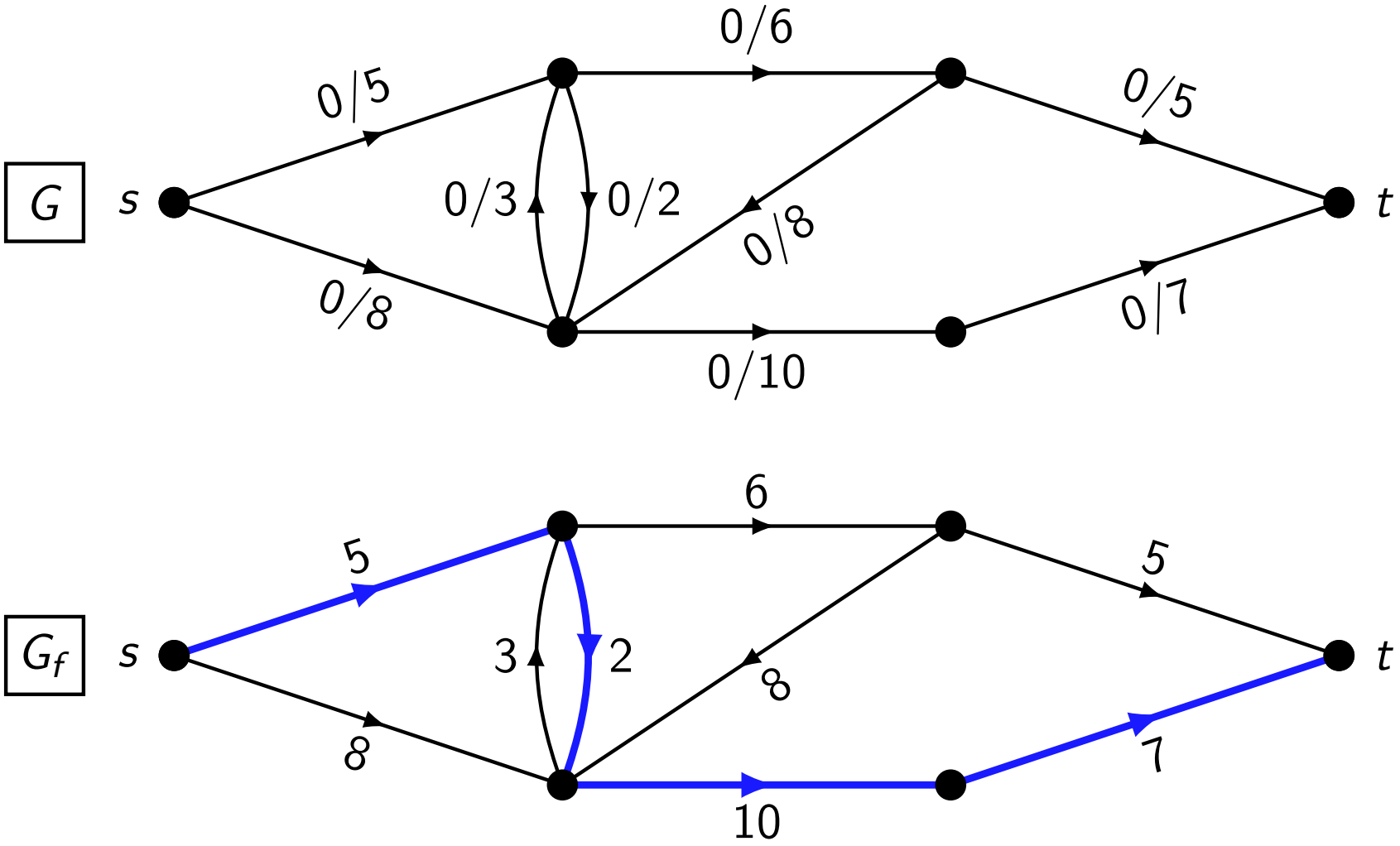

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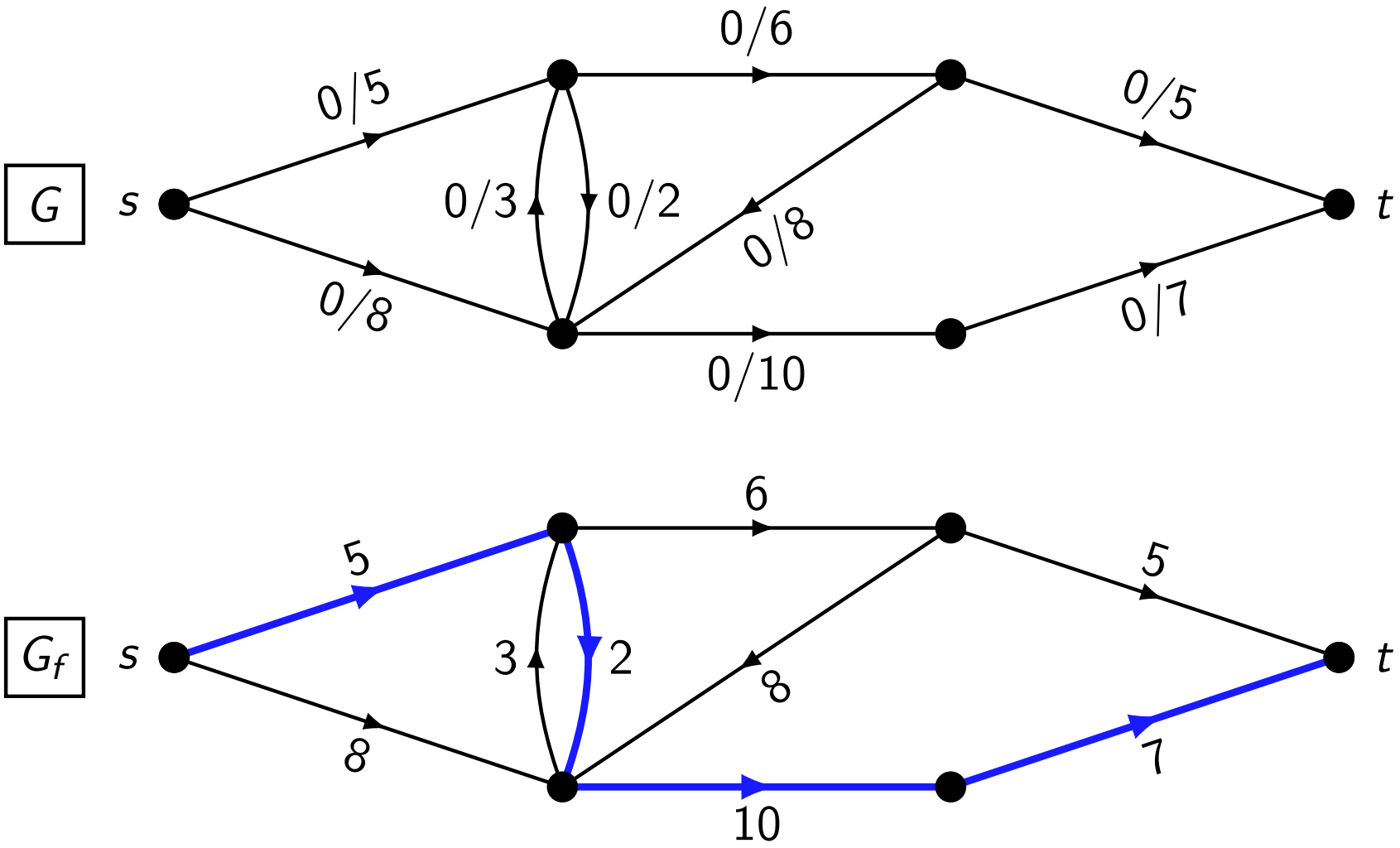

Push flow along the path. (This path has residual capacity 2.)

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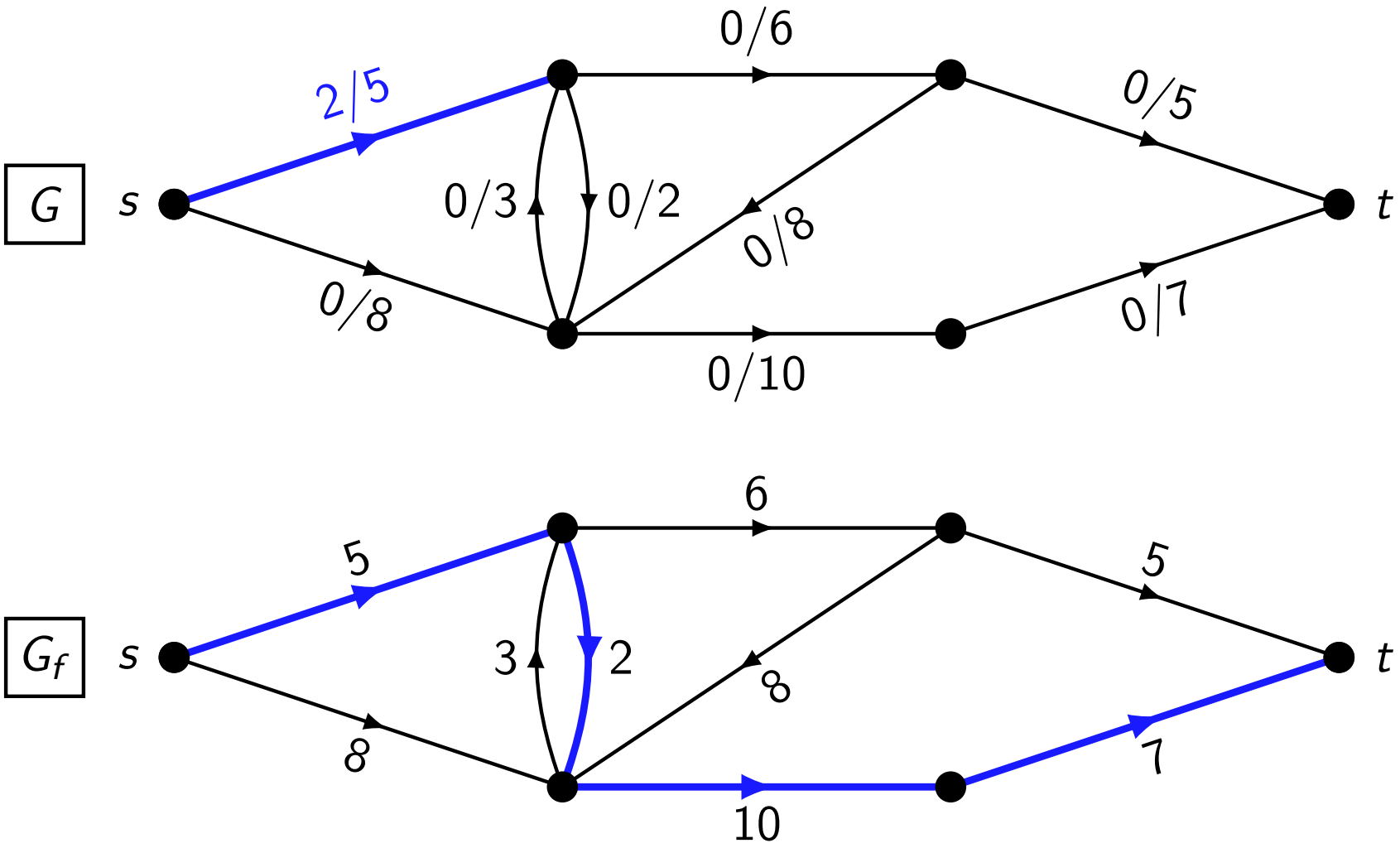

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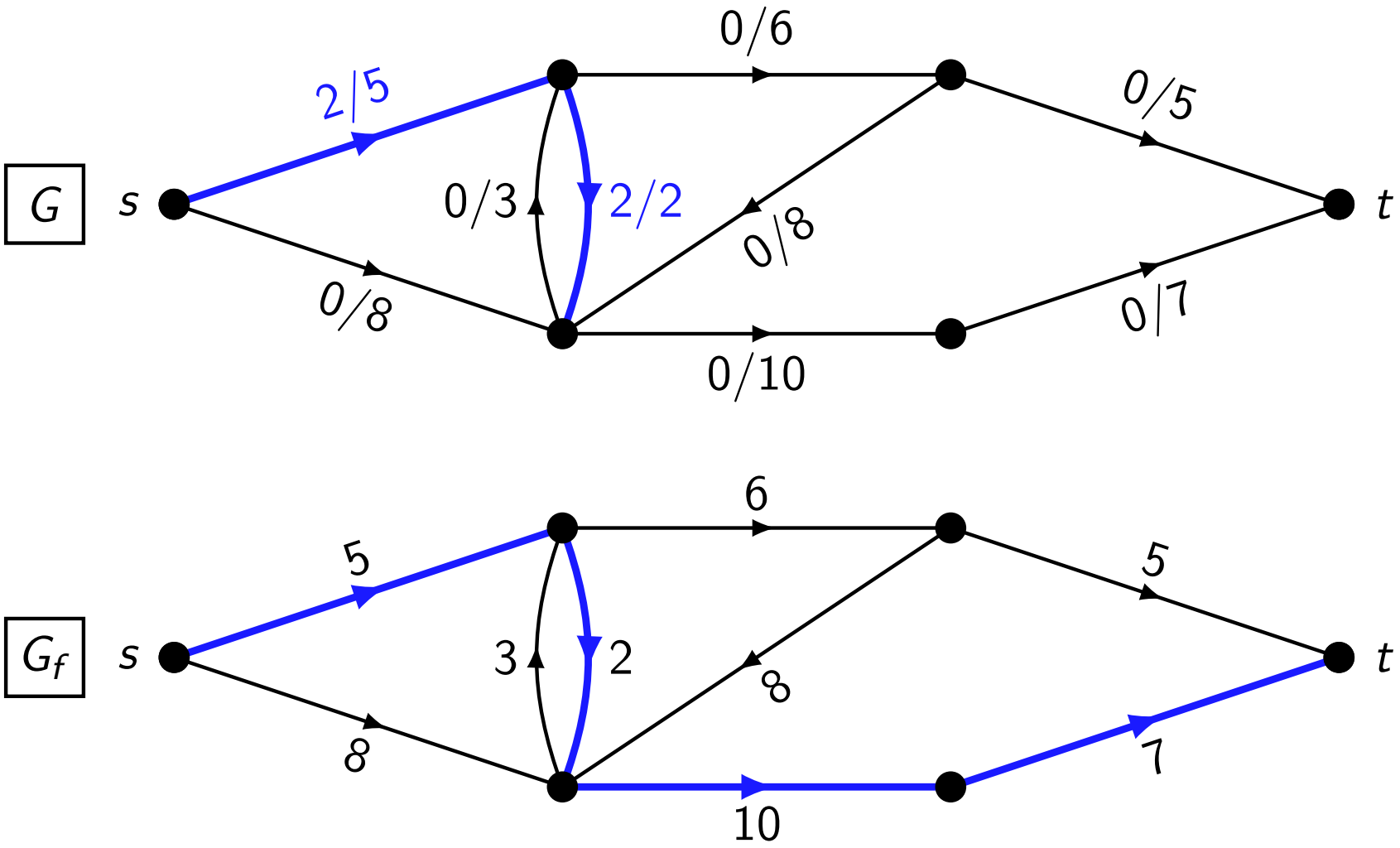

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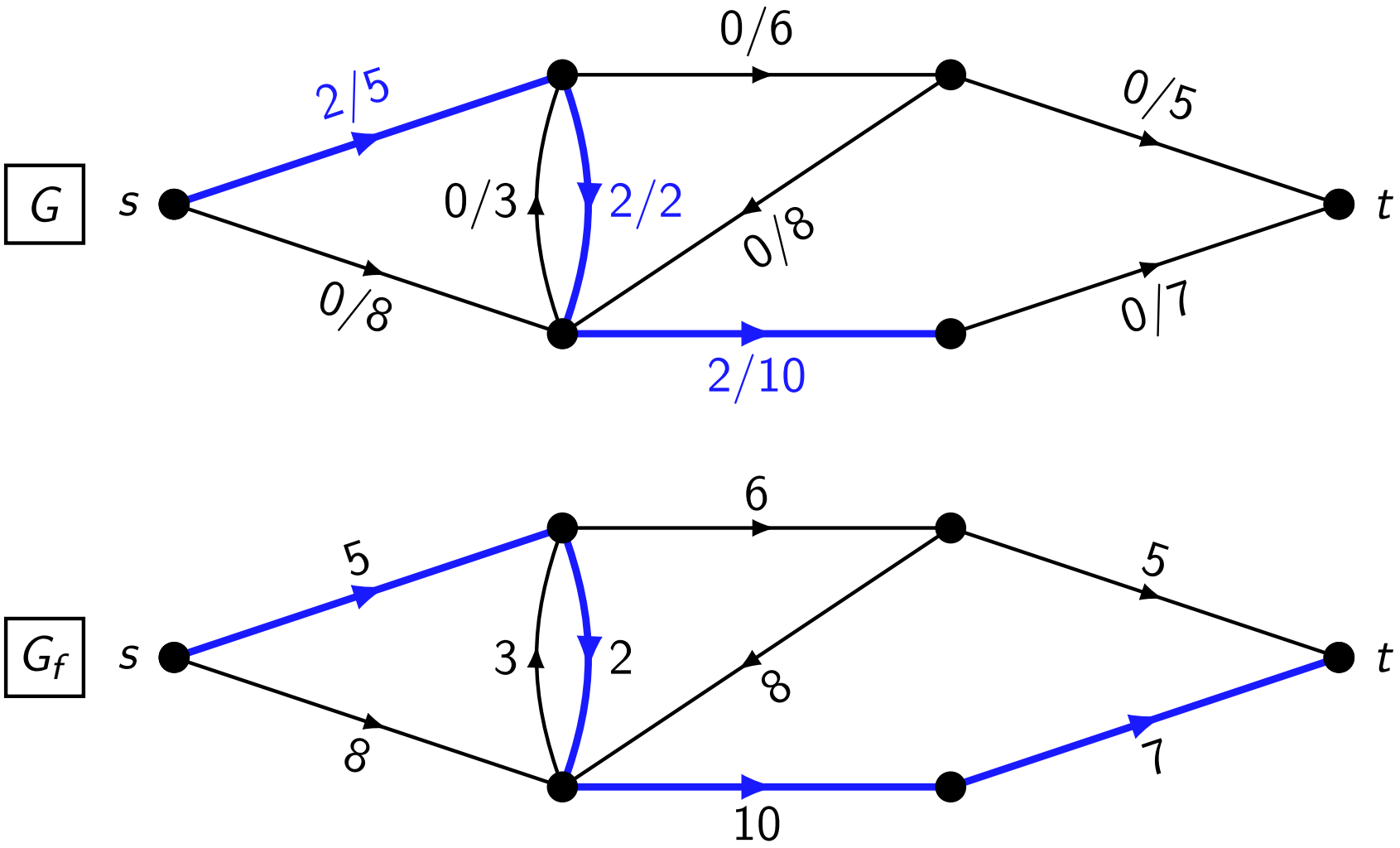

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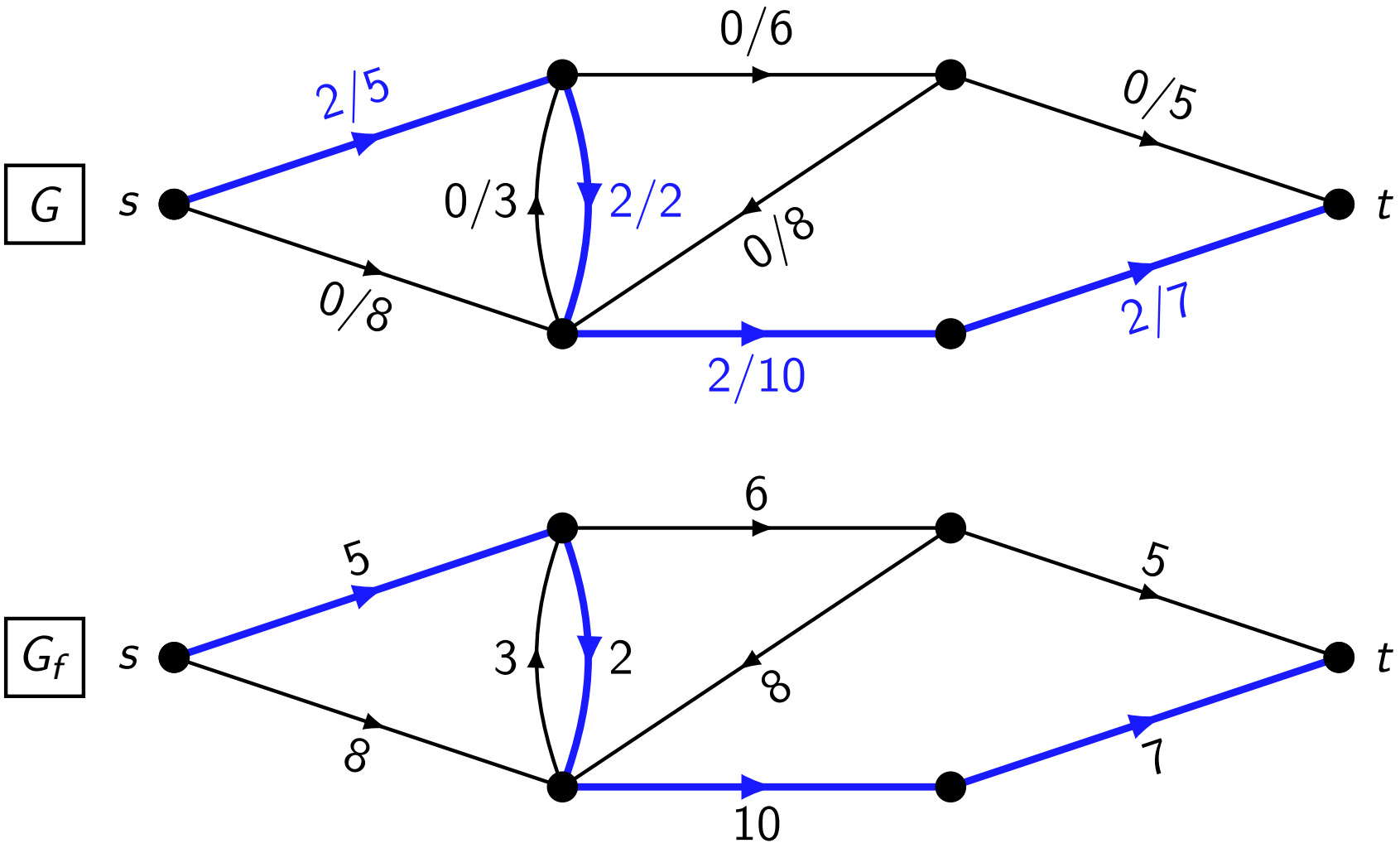

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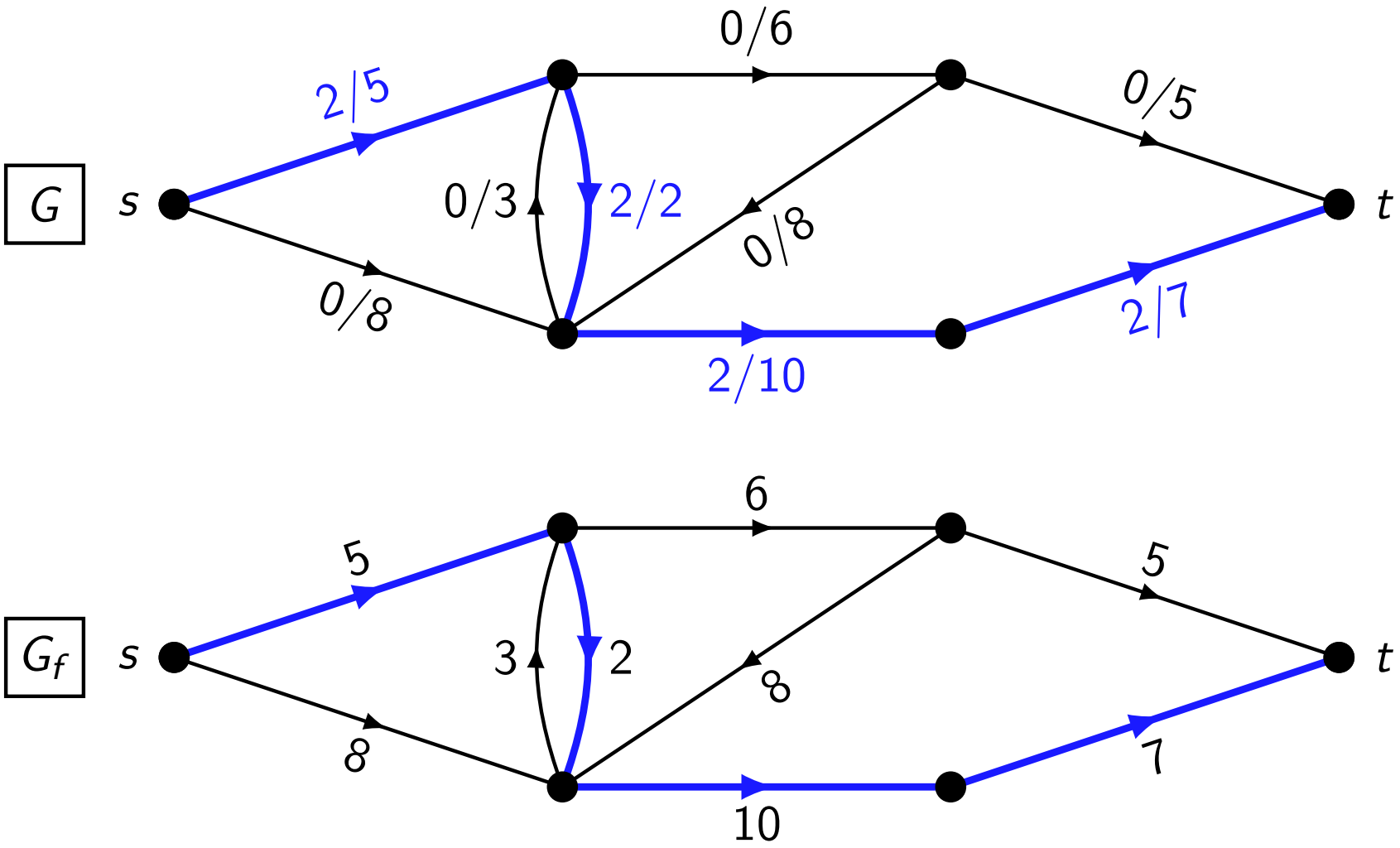

Update G_f along the path.

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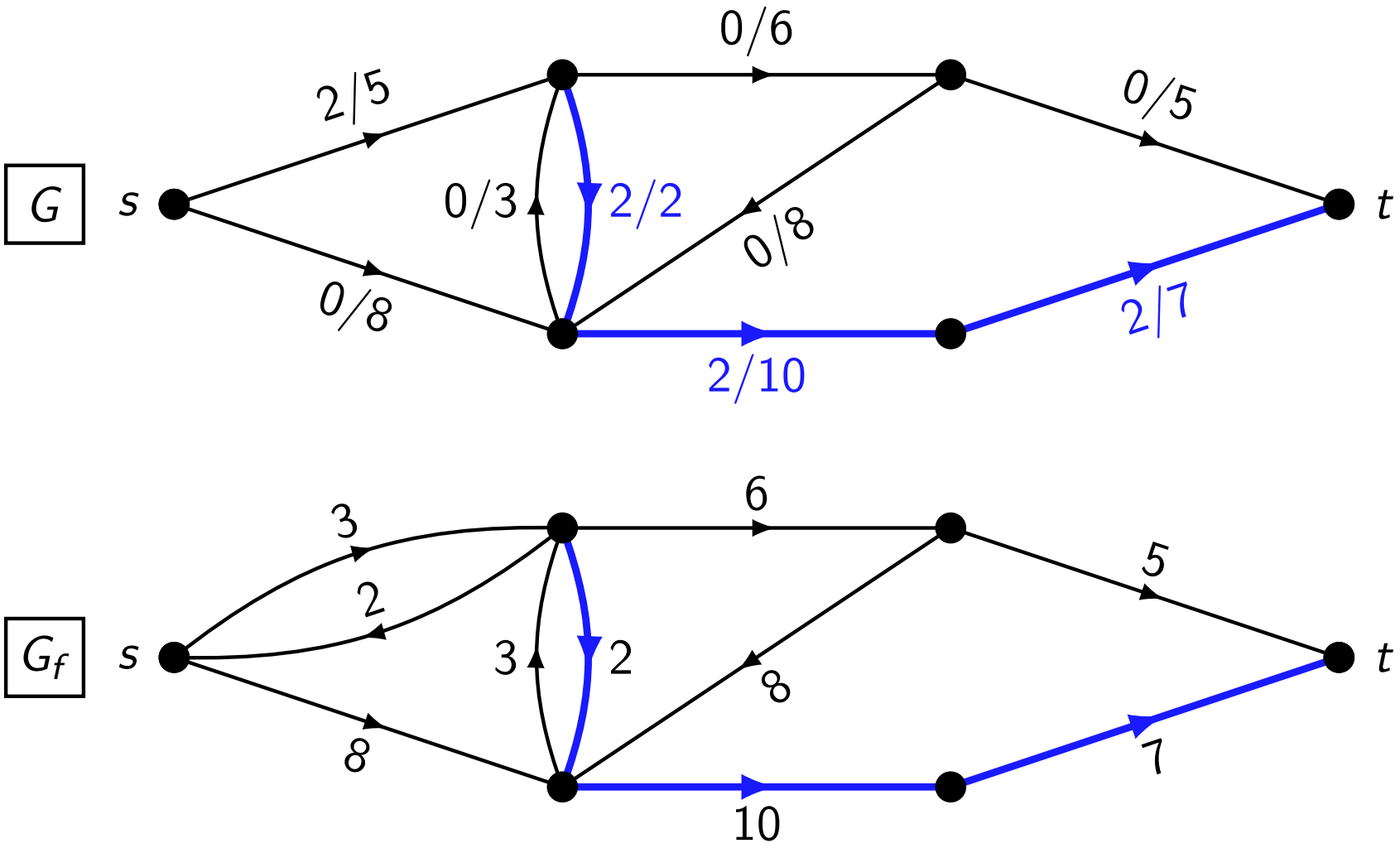

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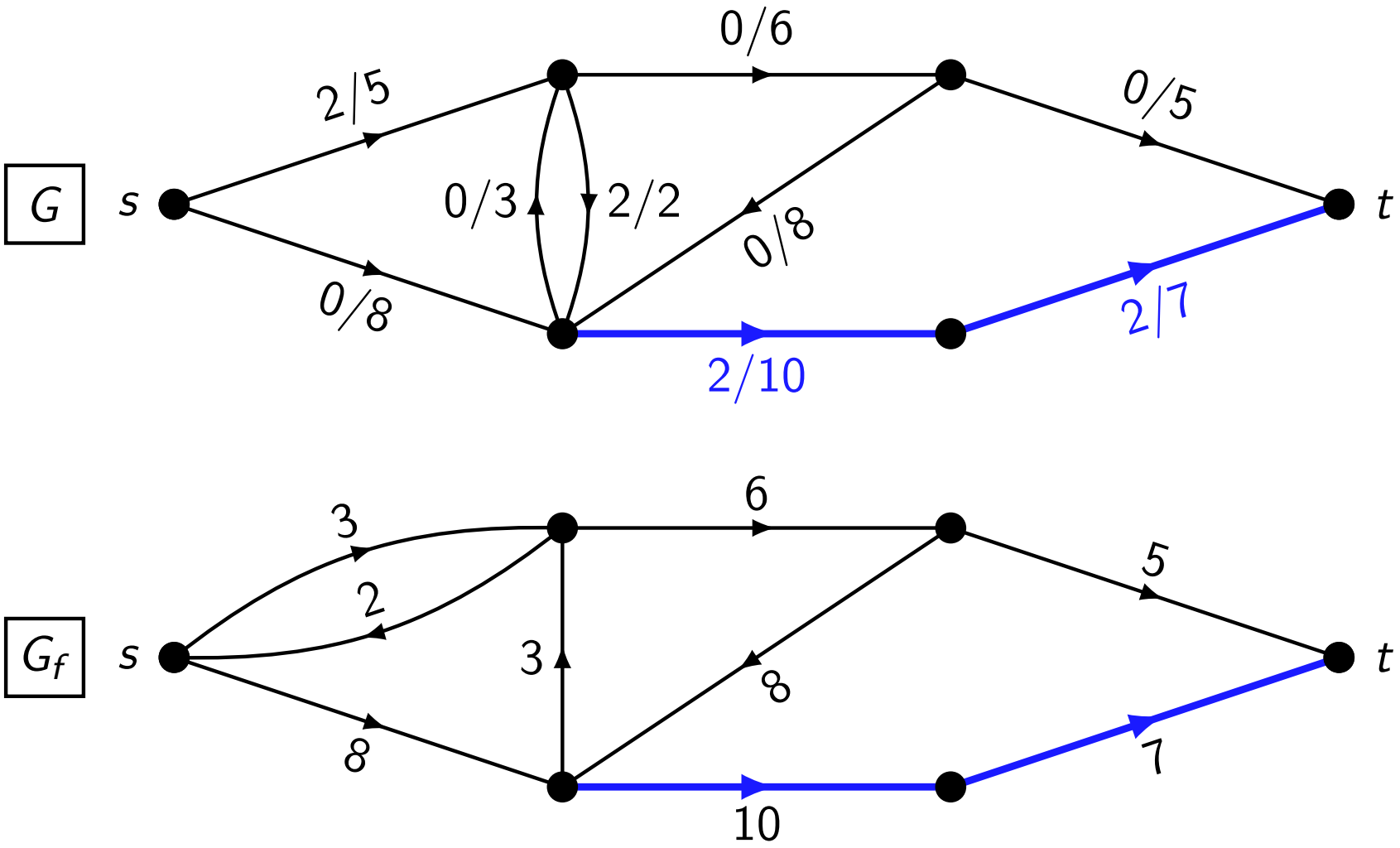

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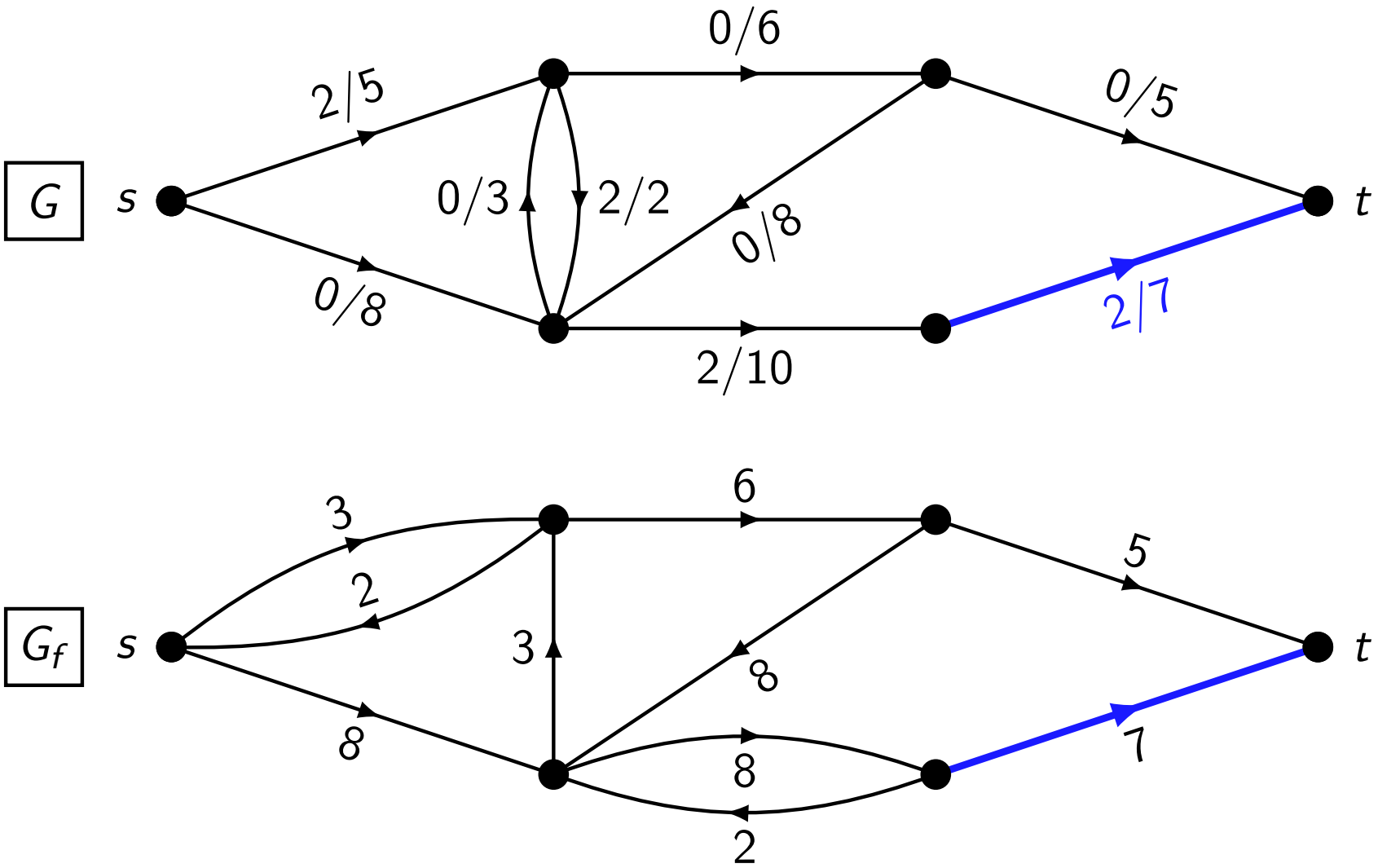

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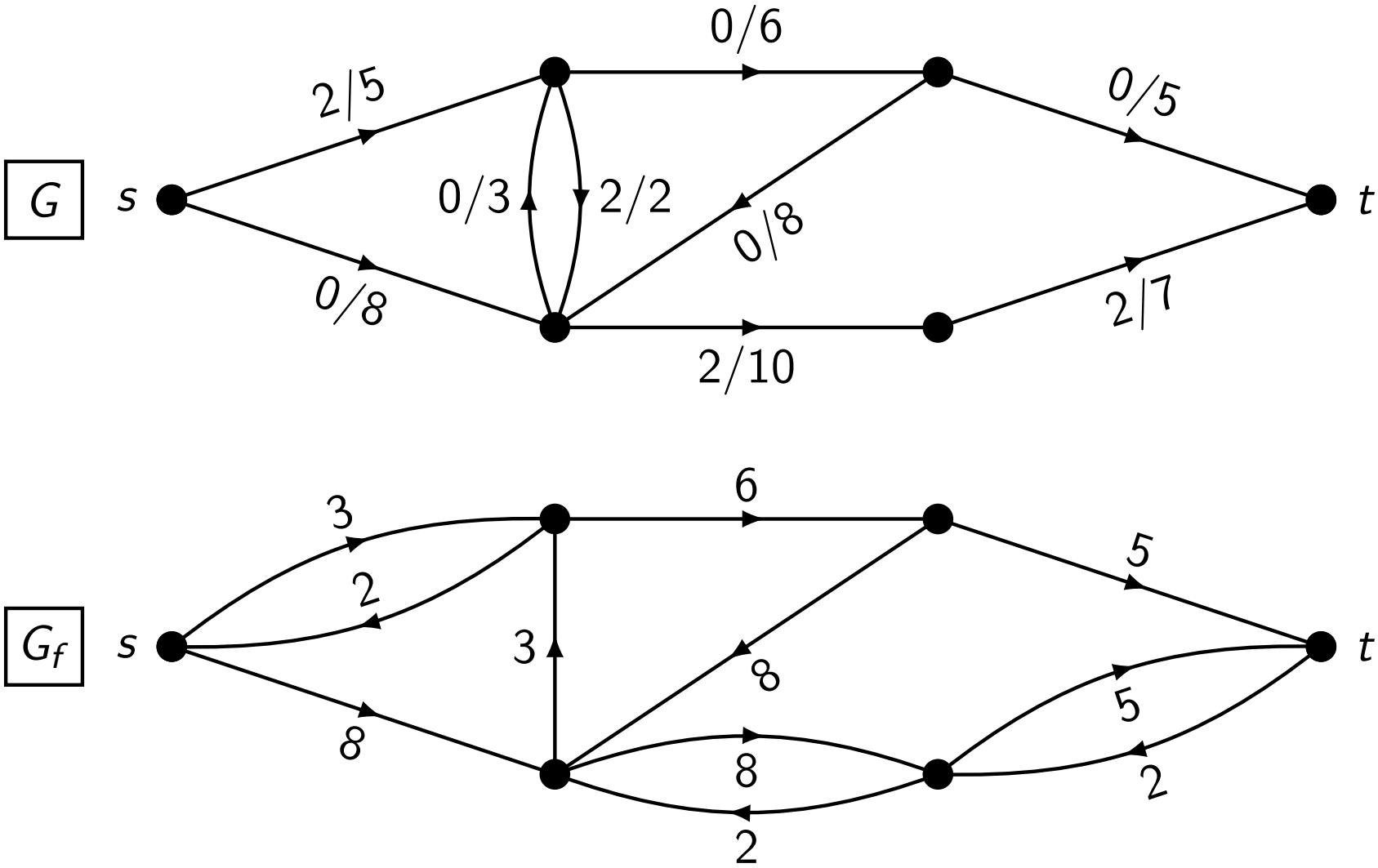

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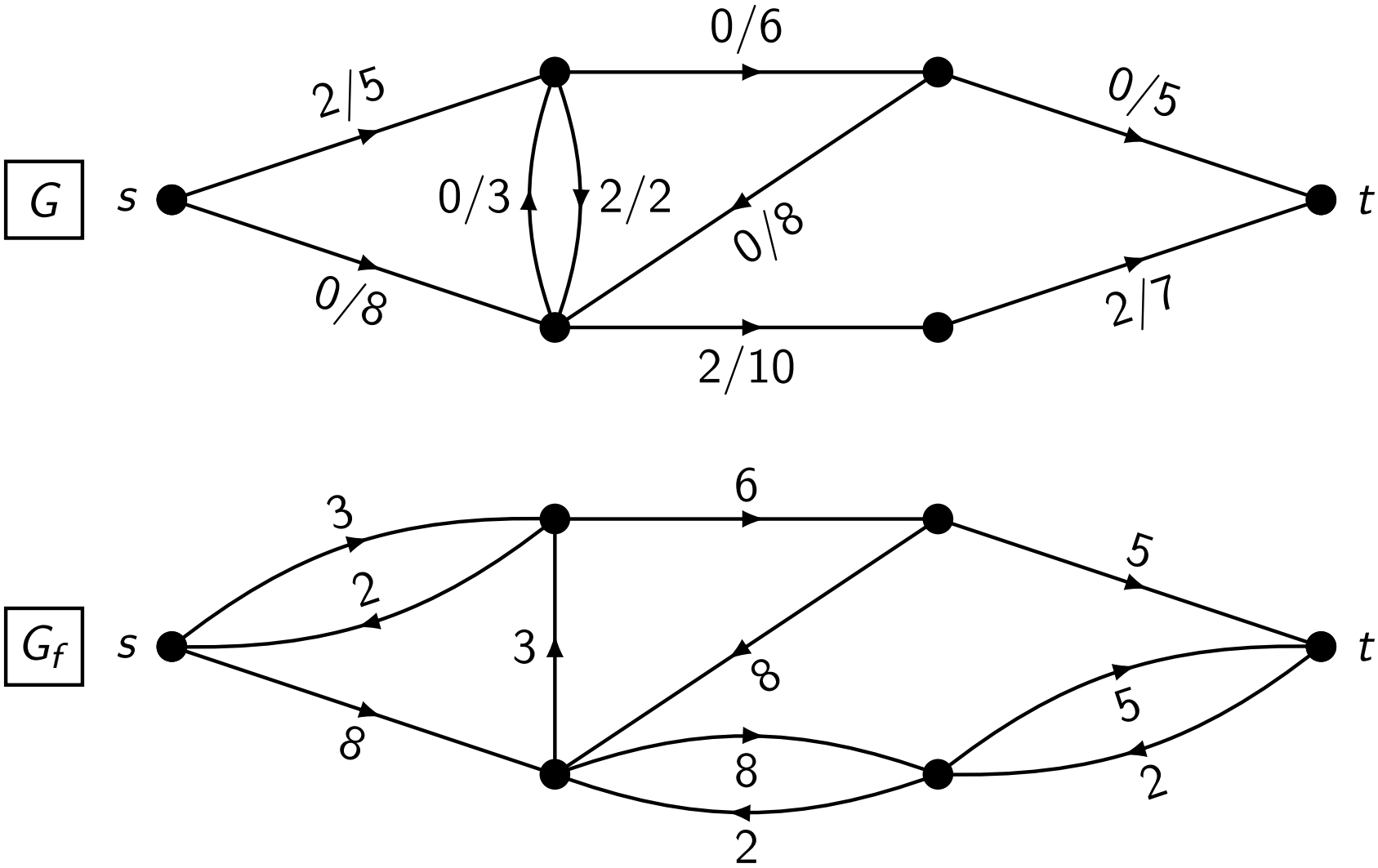

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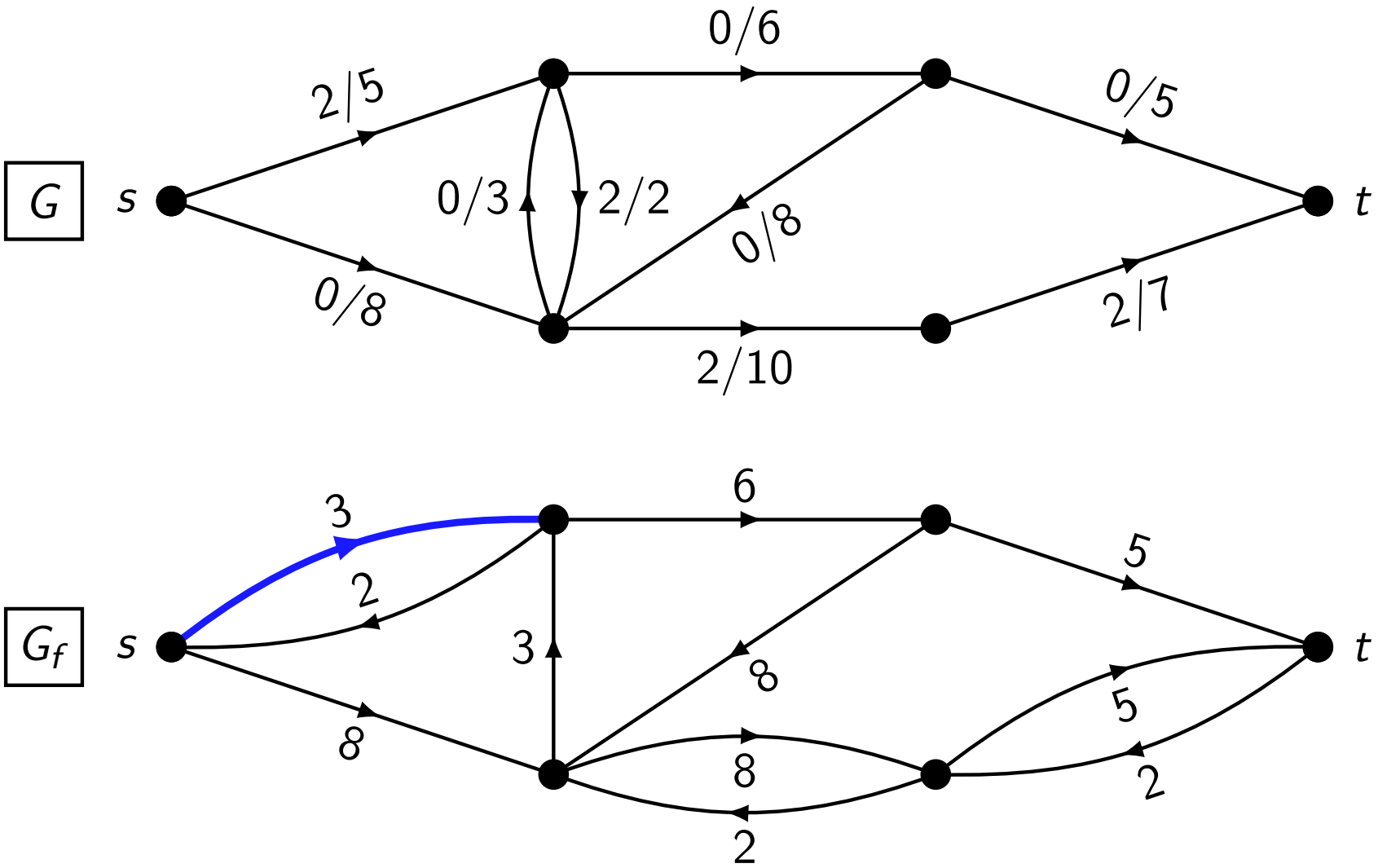

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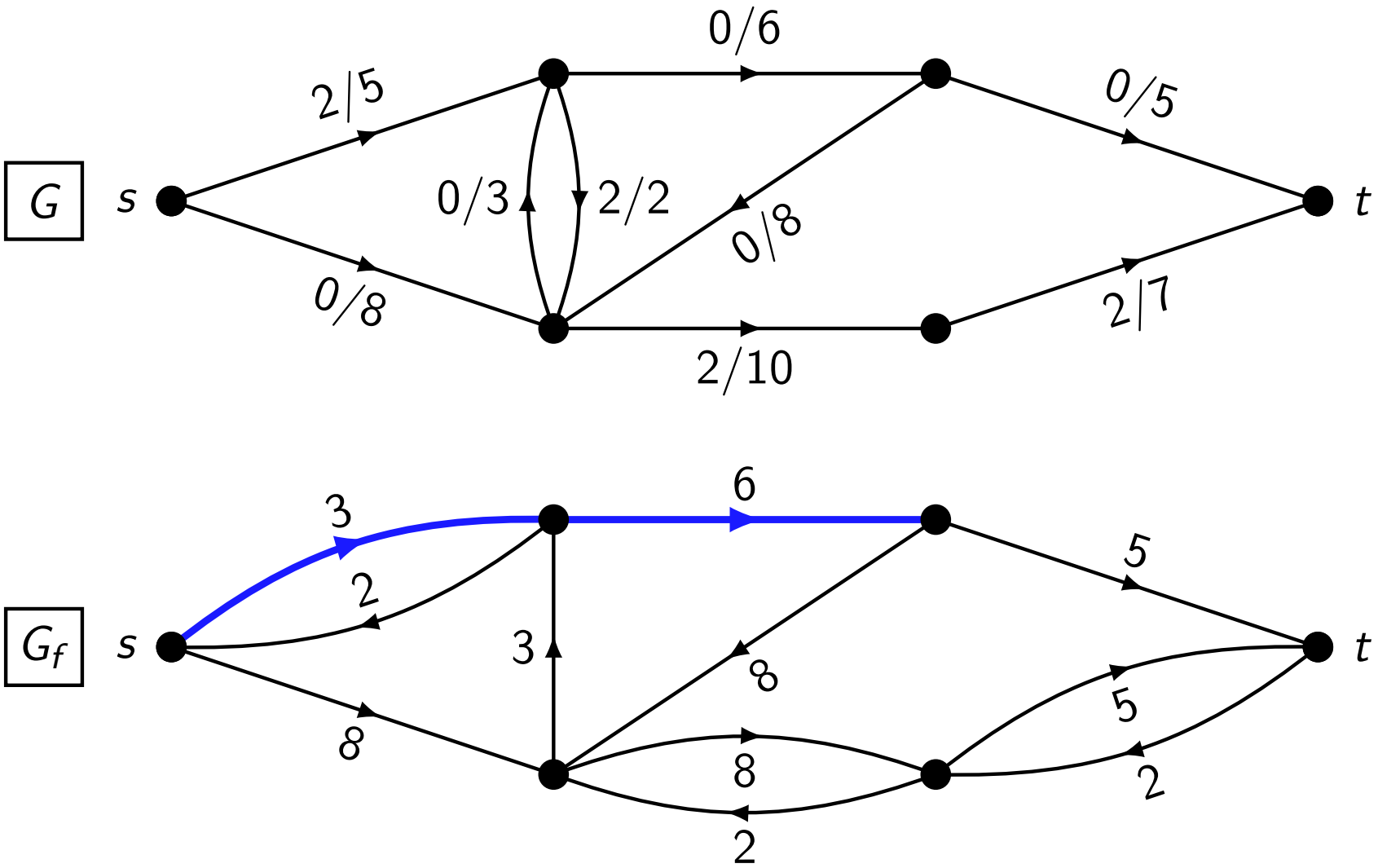

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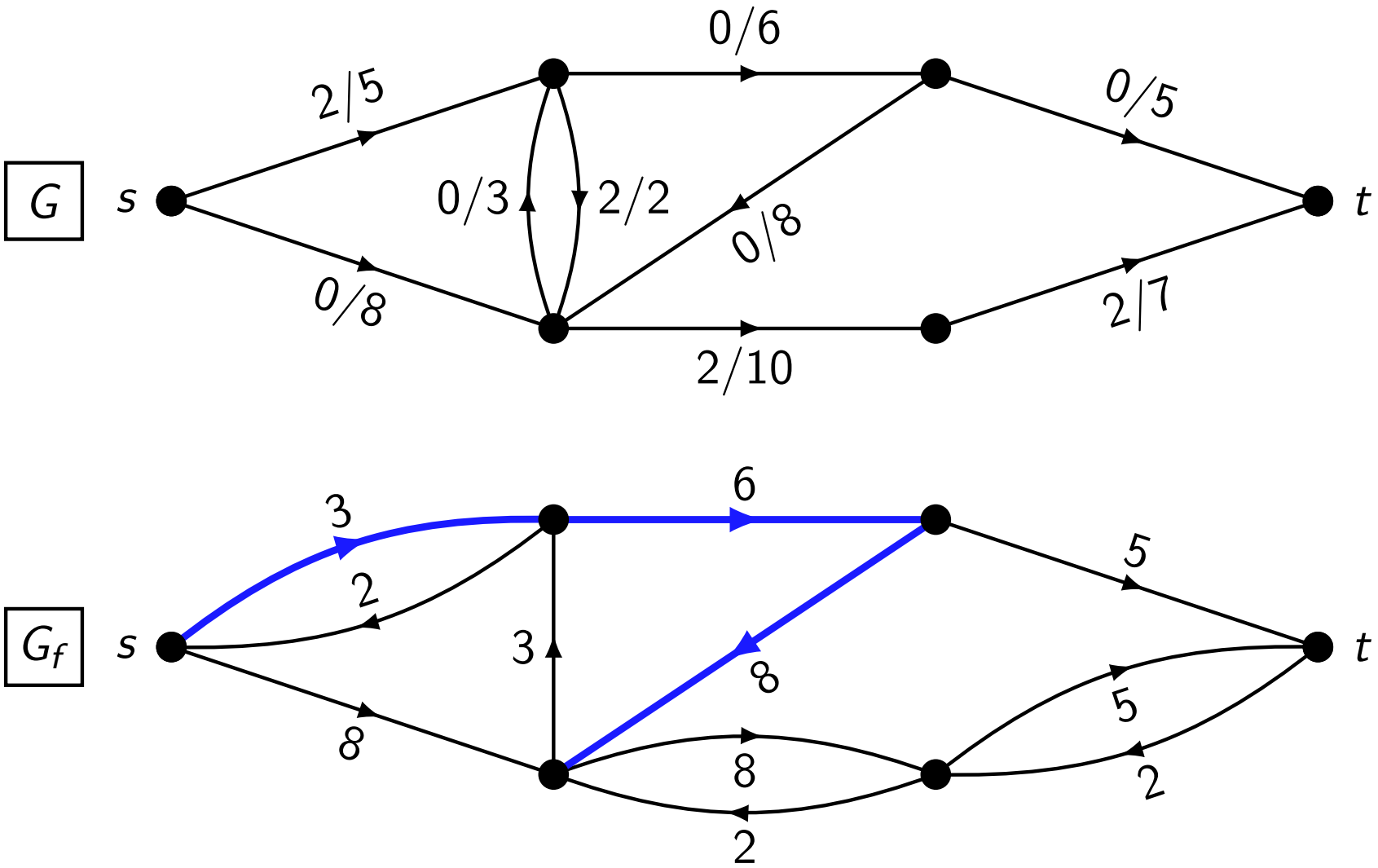

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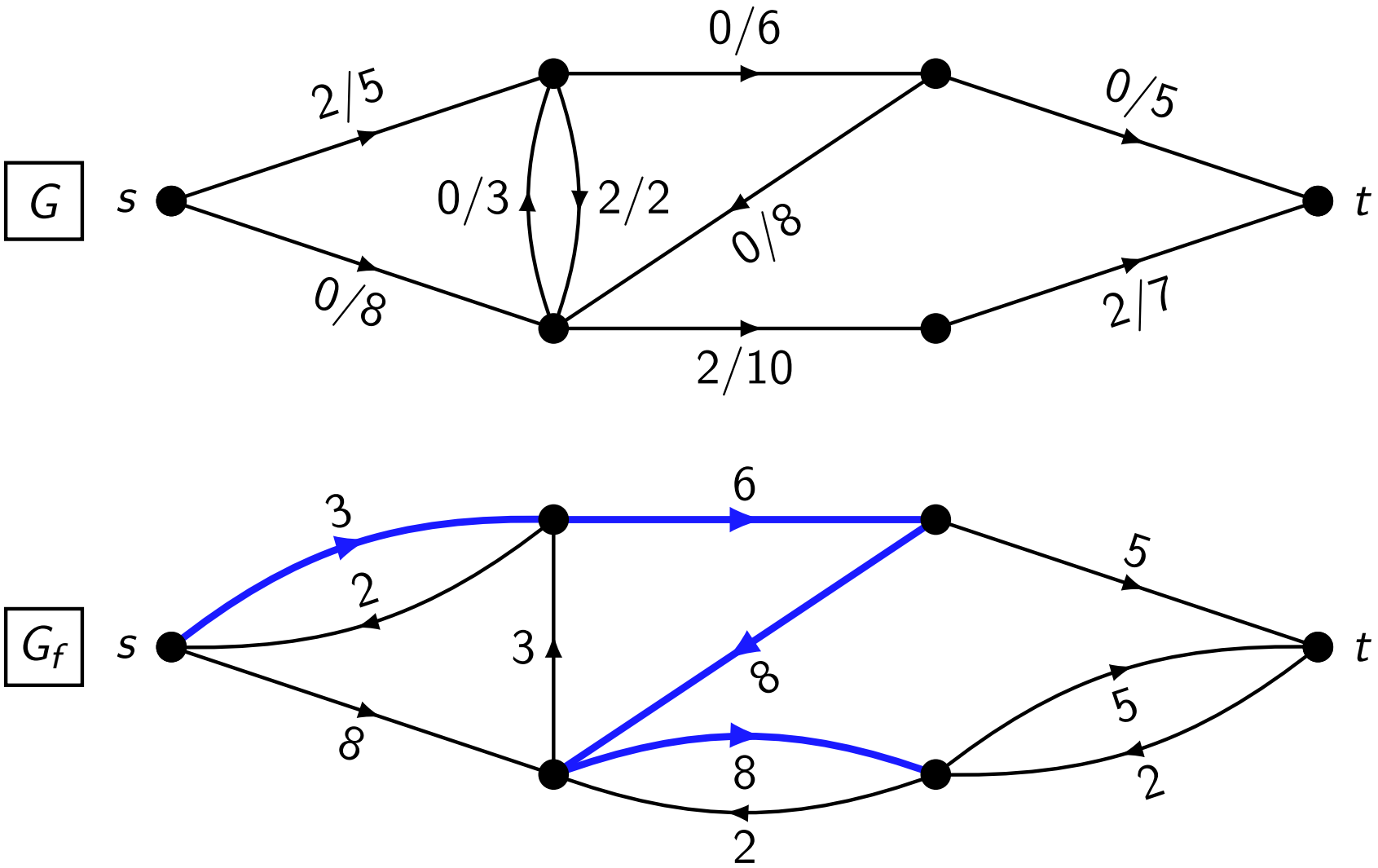

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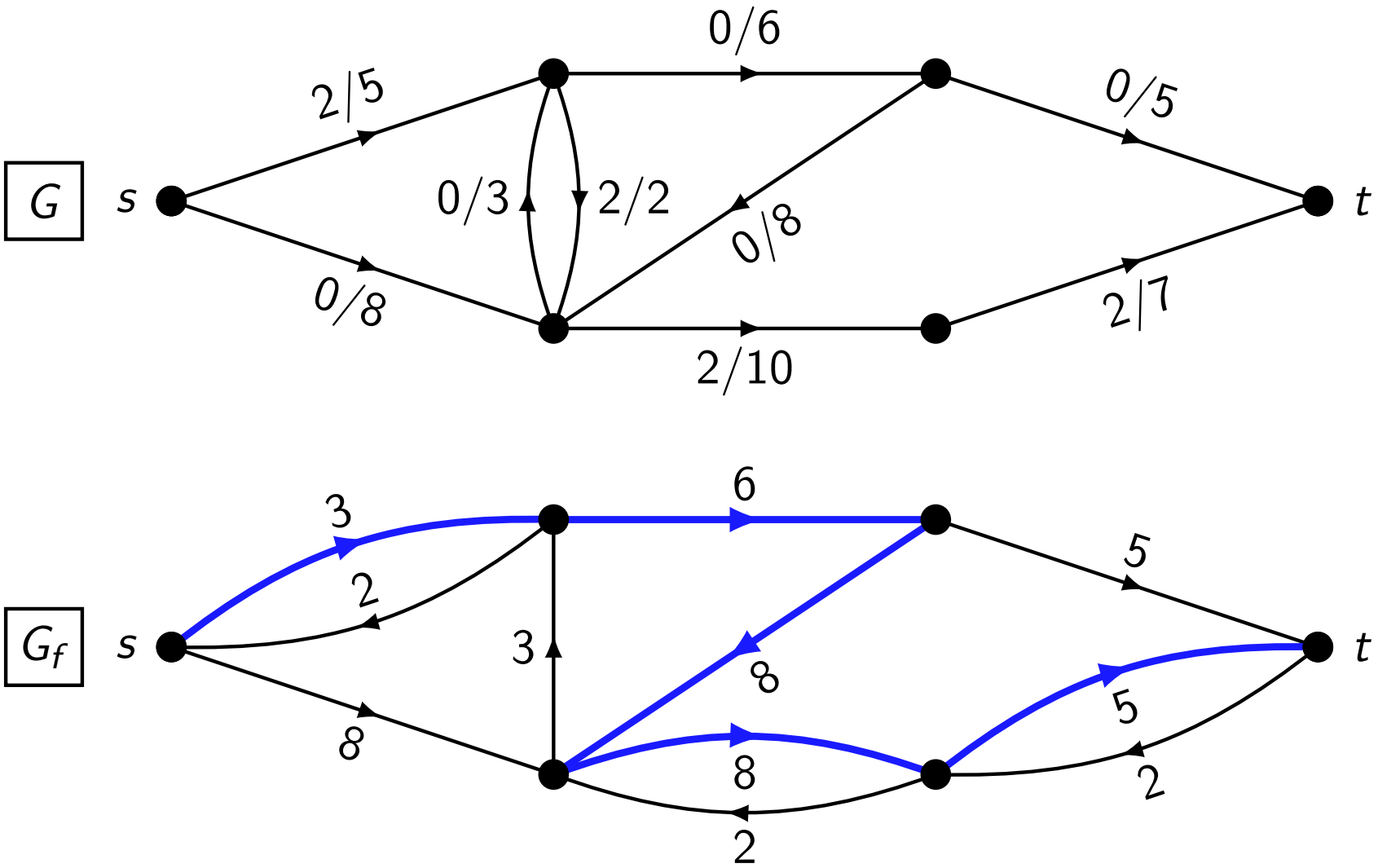

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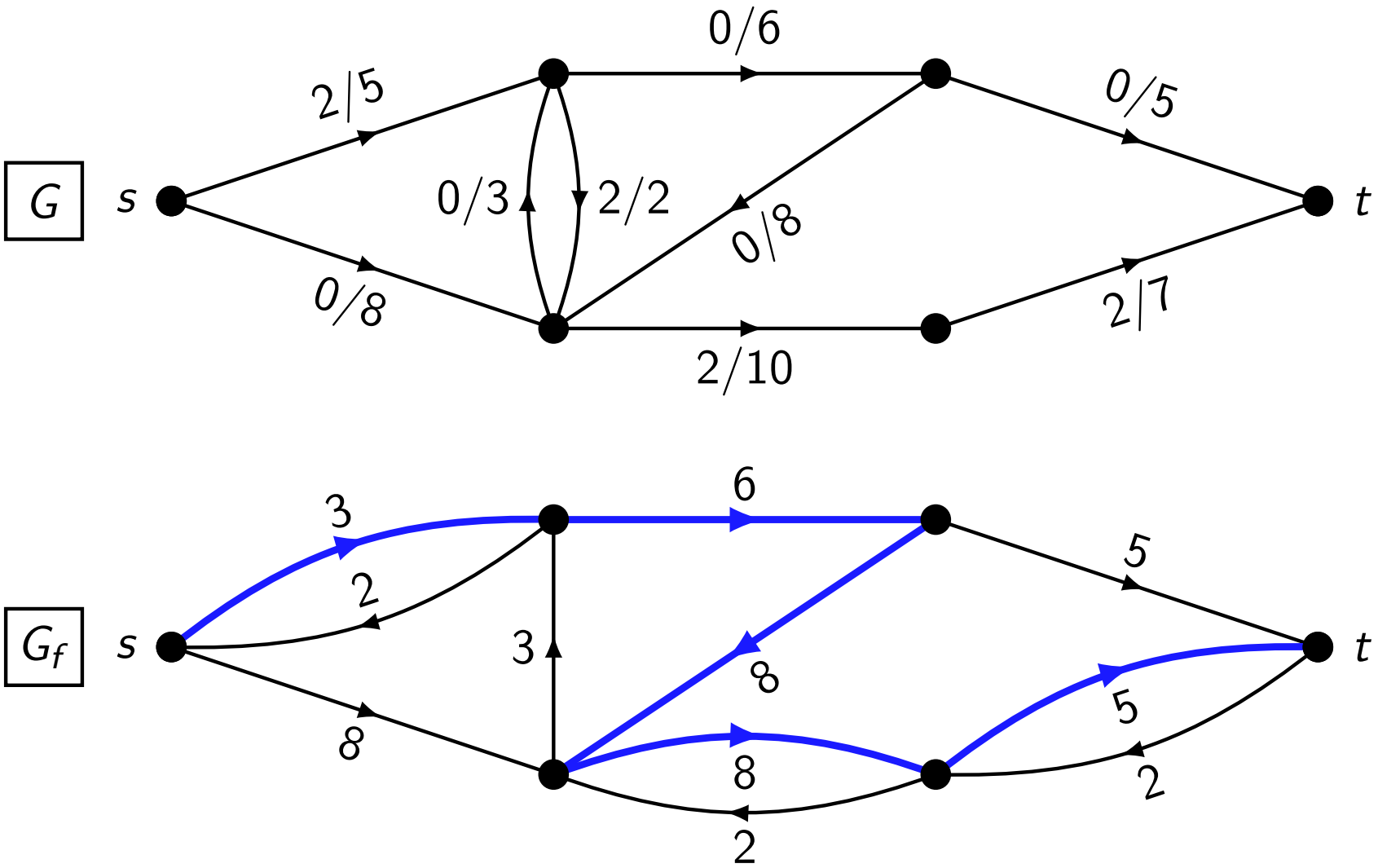

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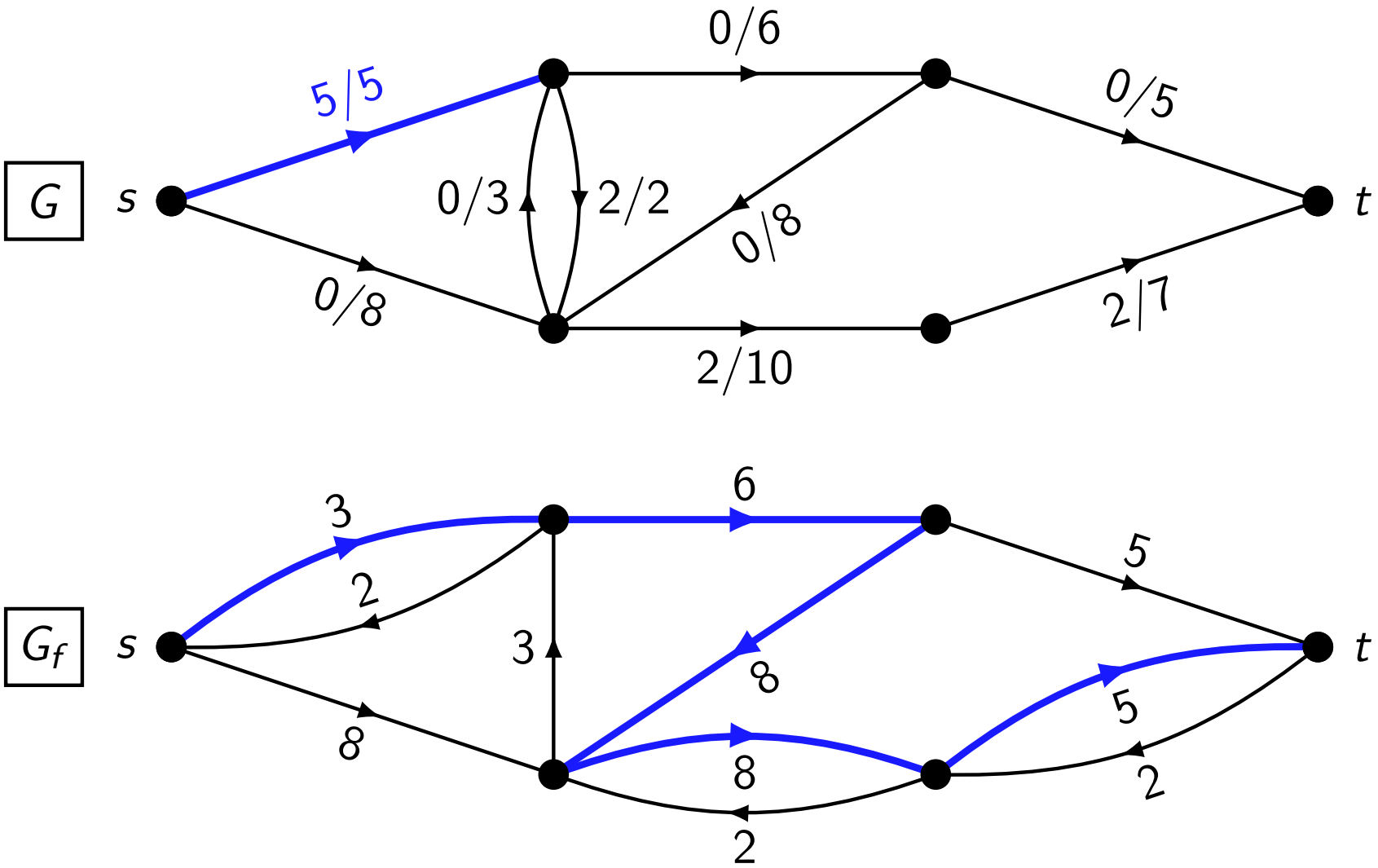

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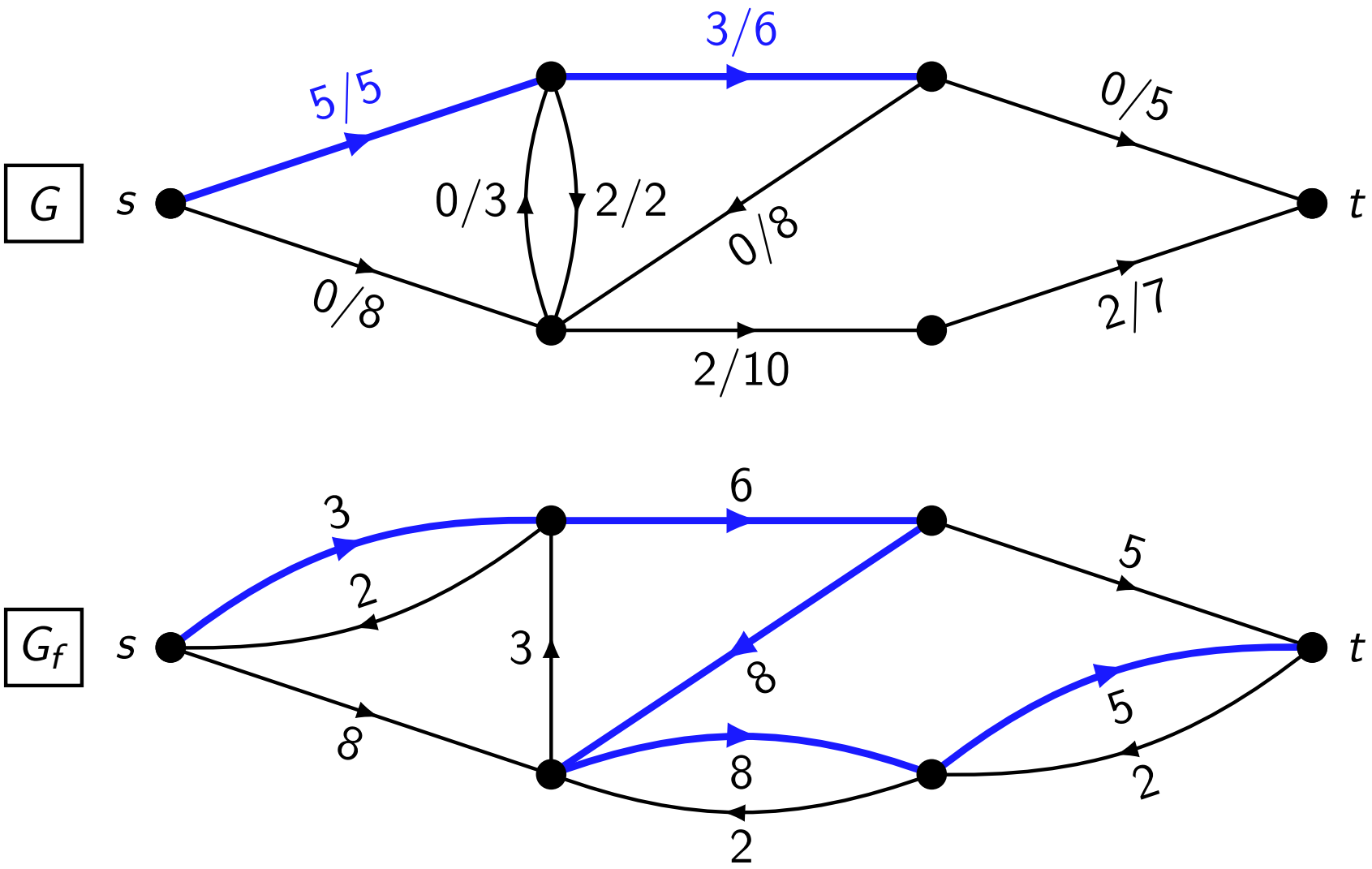

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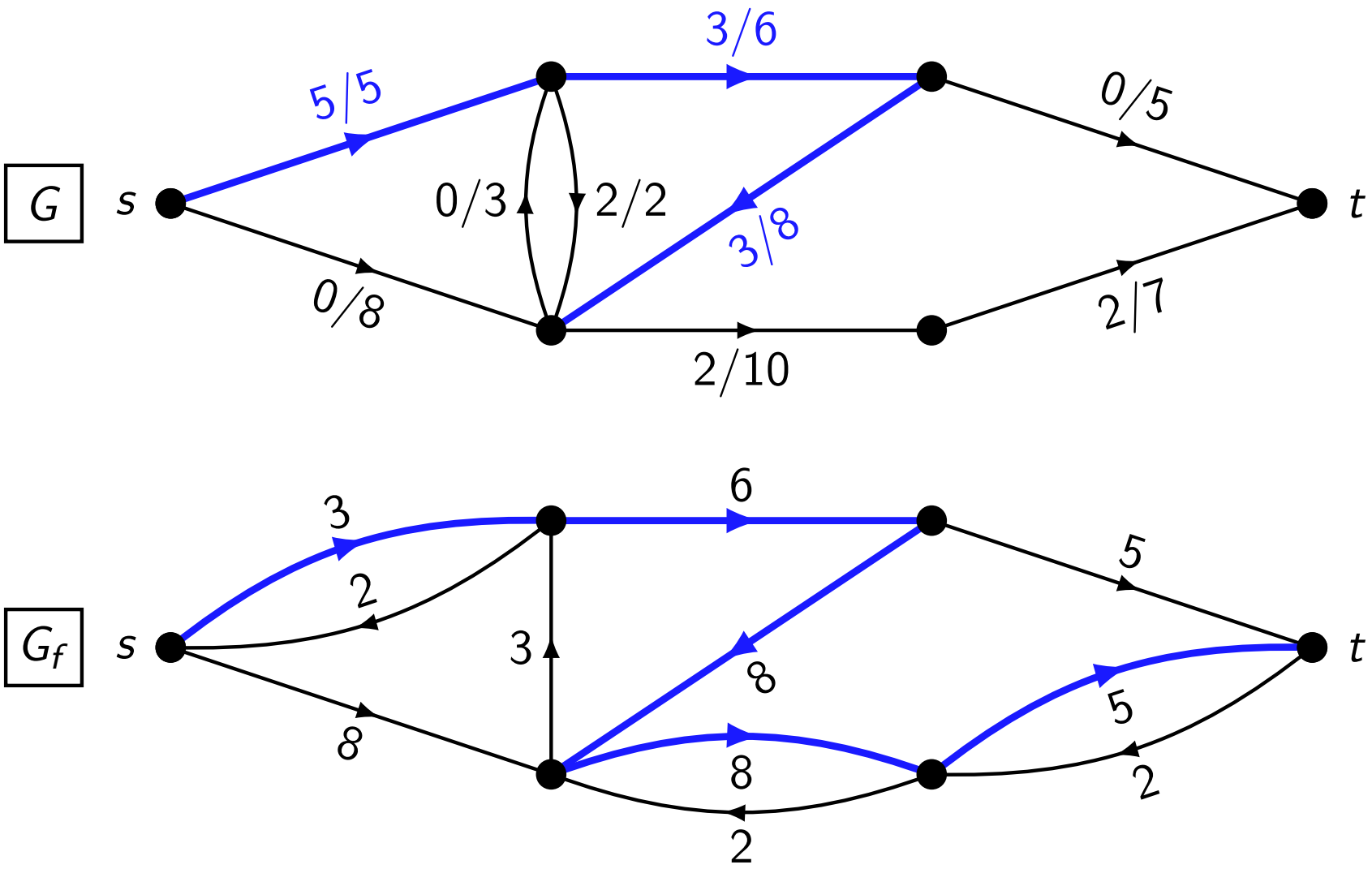

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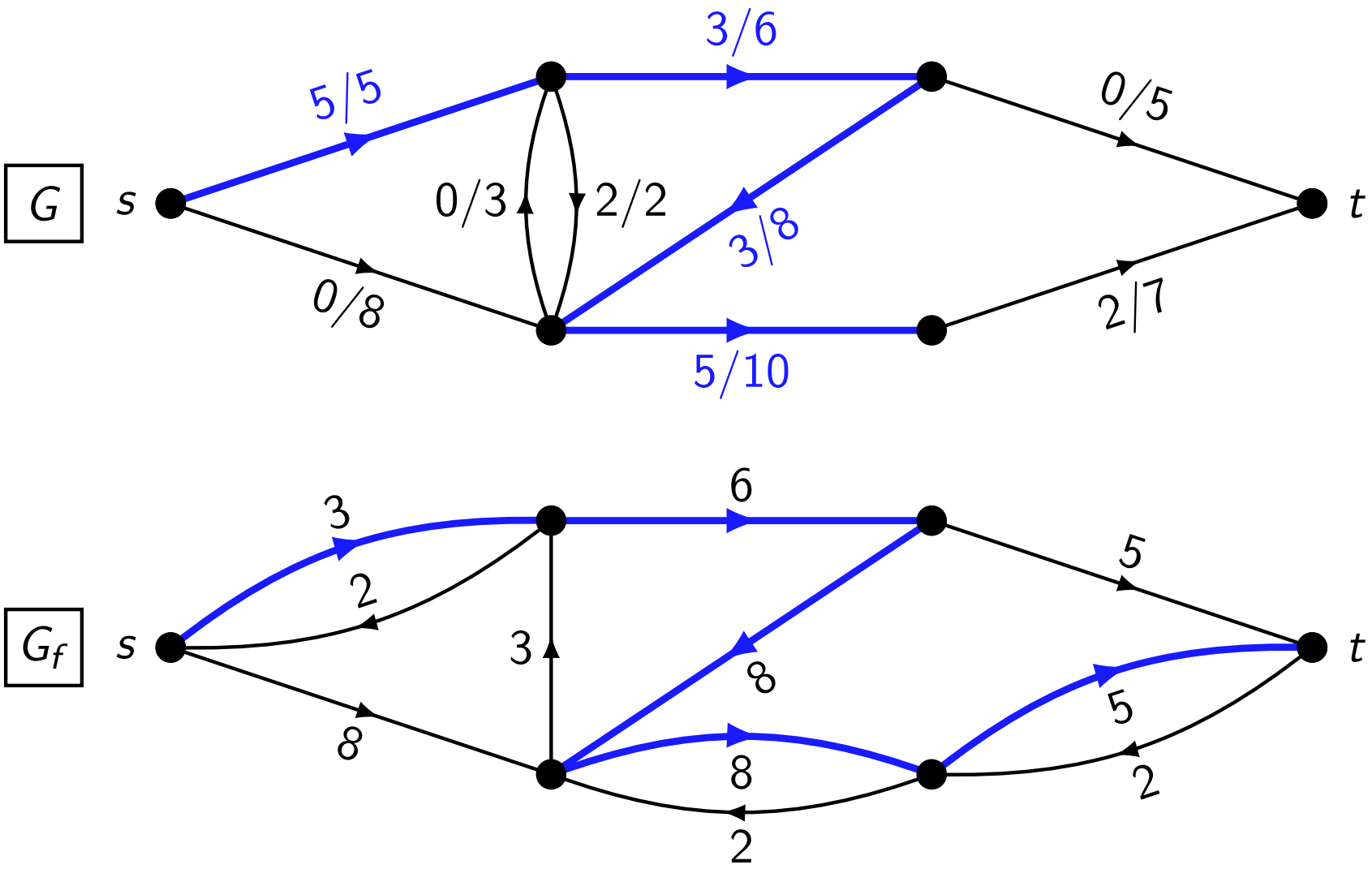

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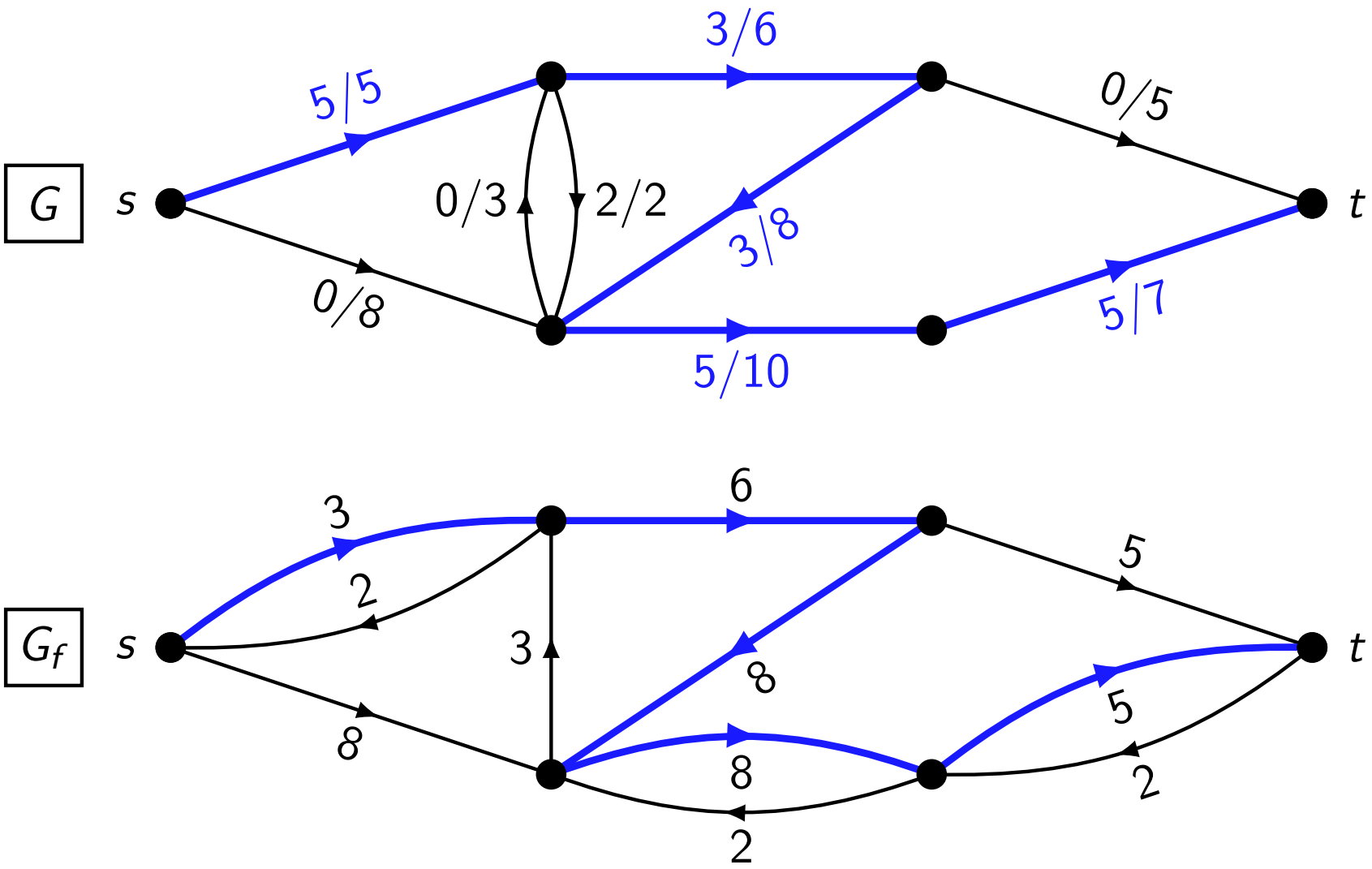

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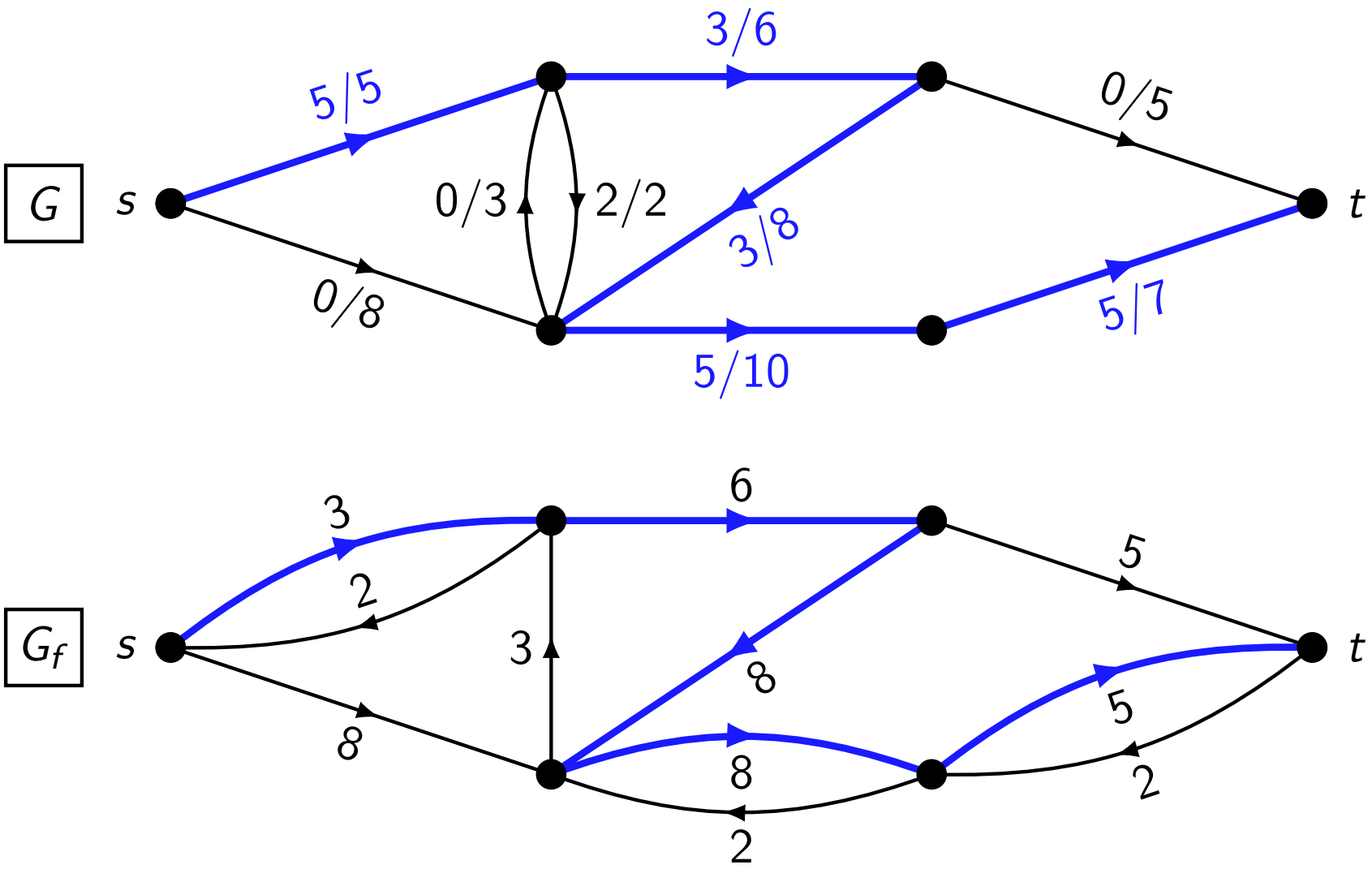

Update G_f along the path.

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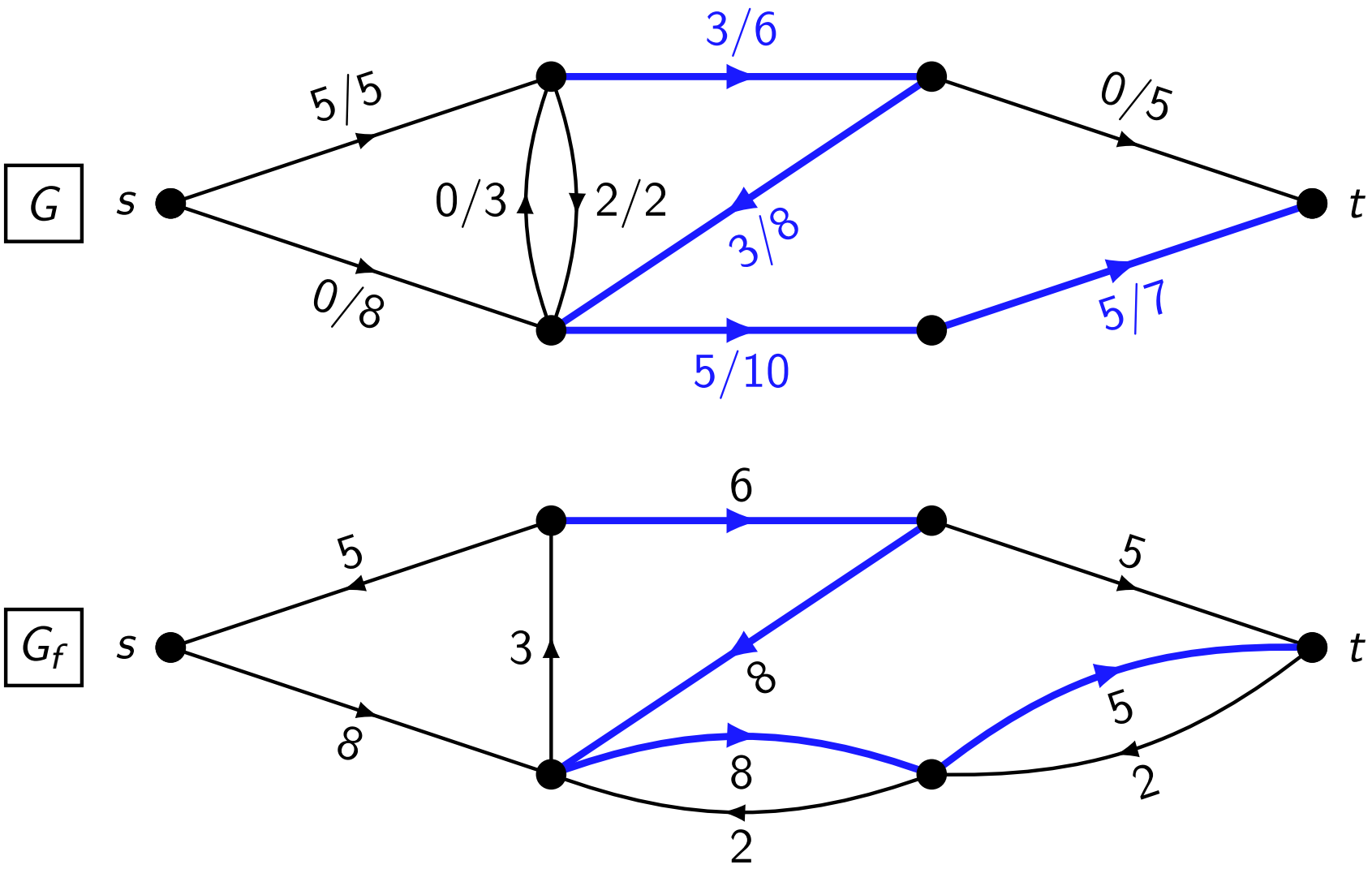

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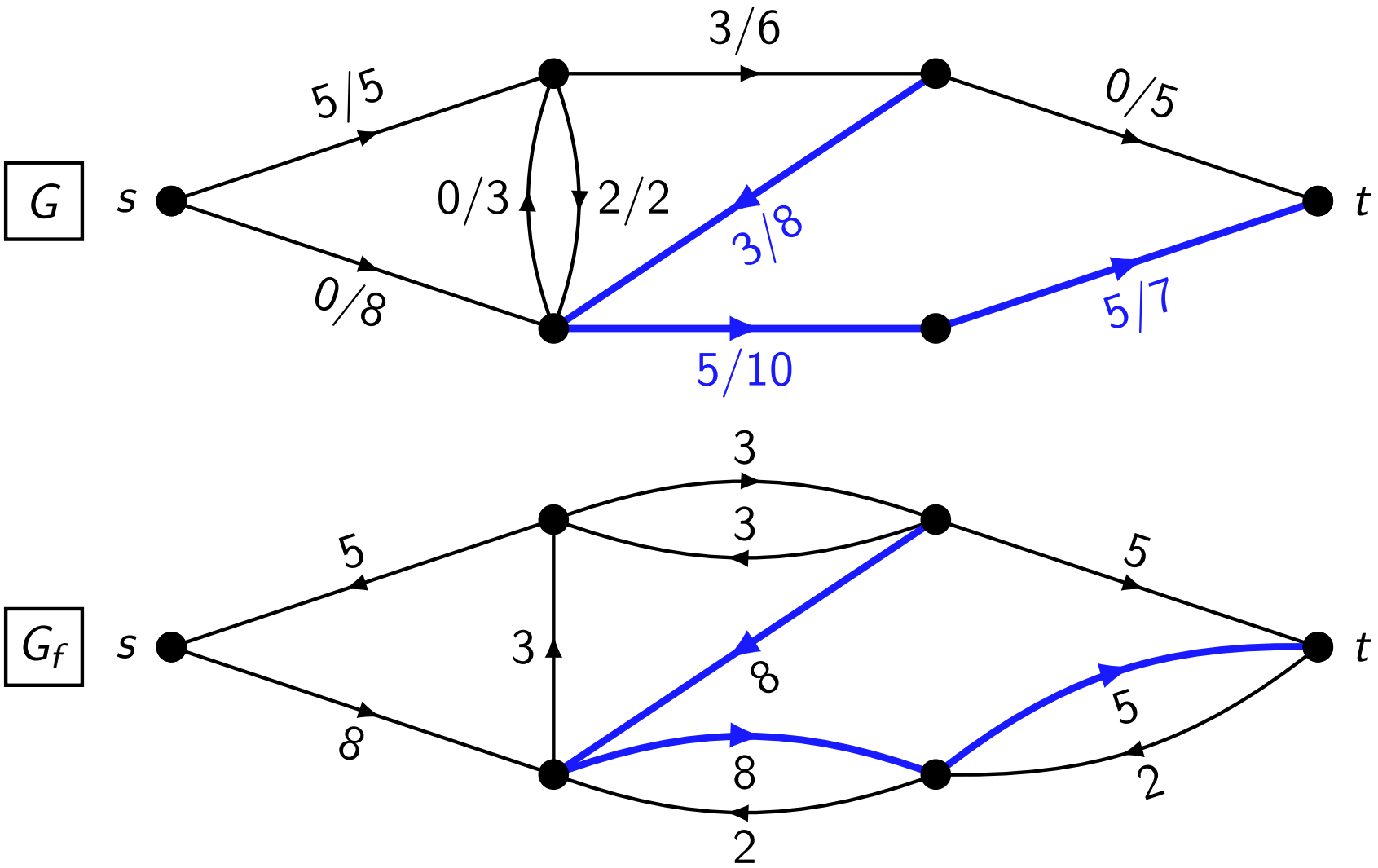

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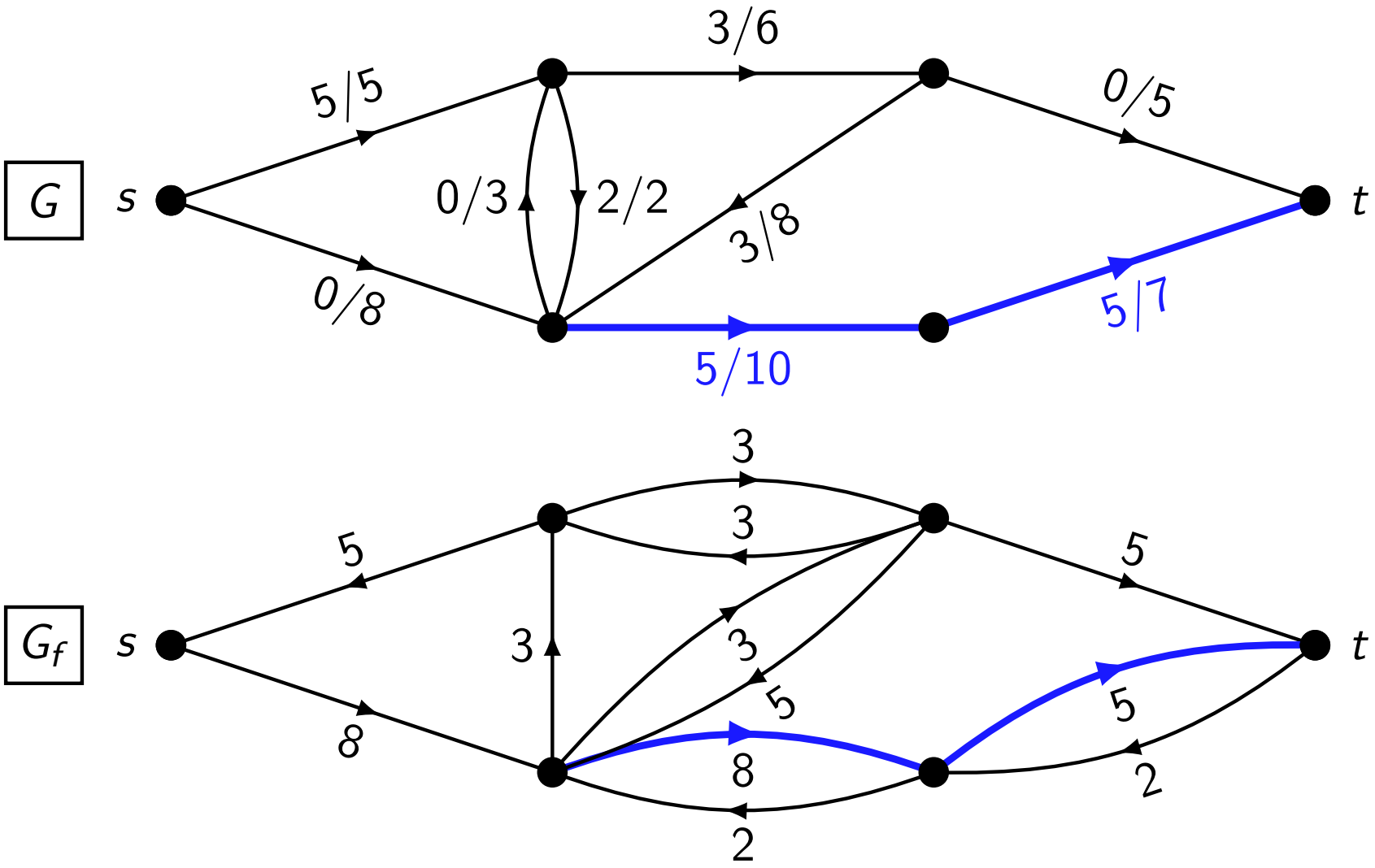

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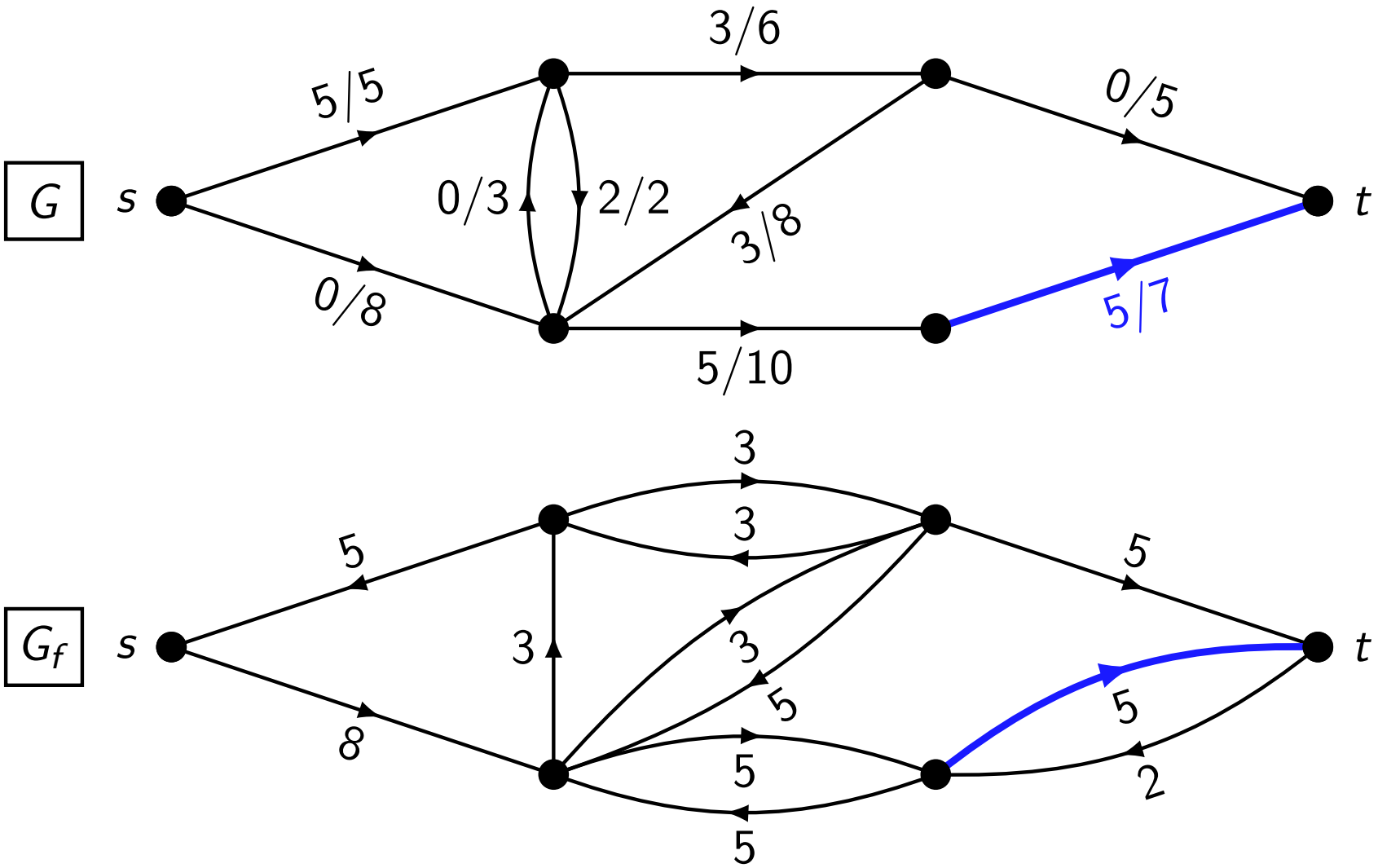

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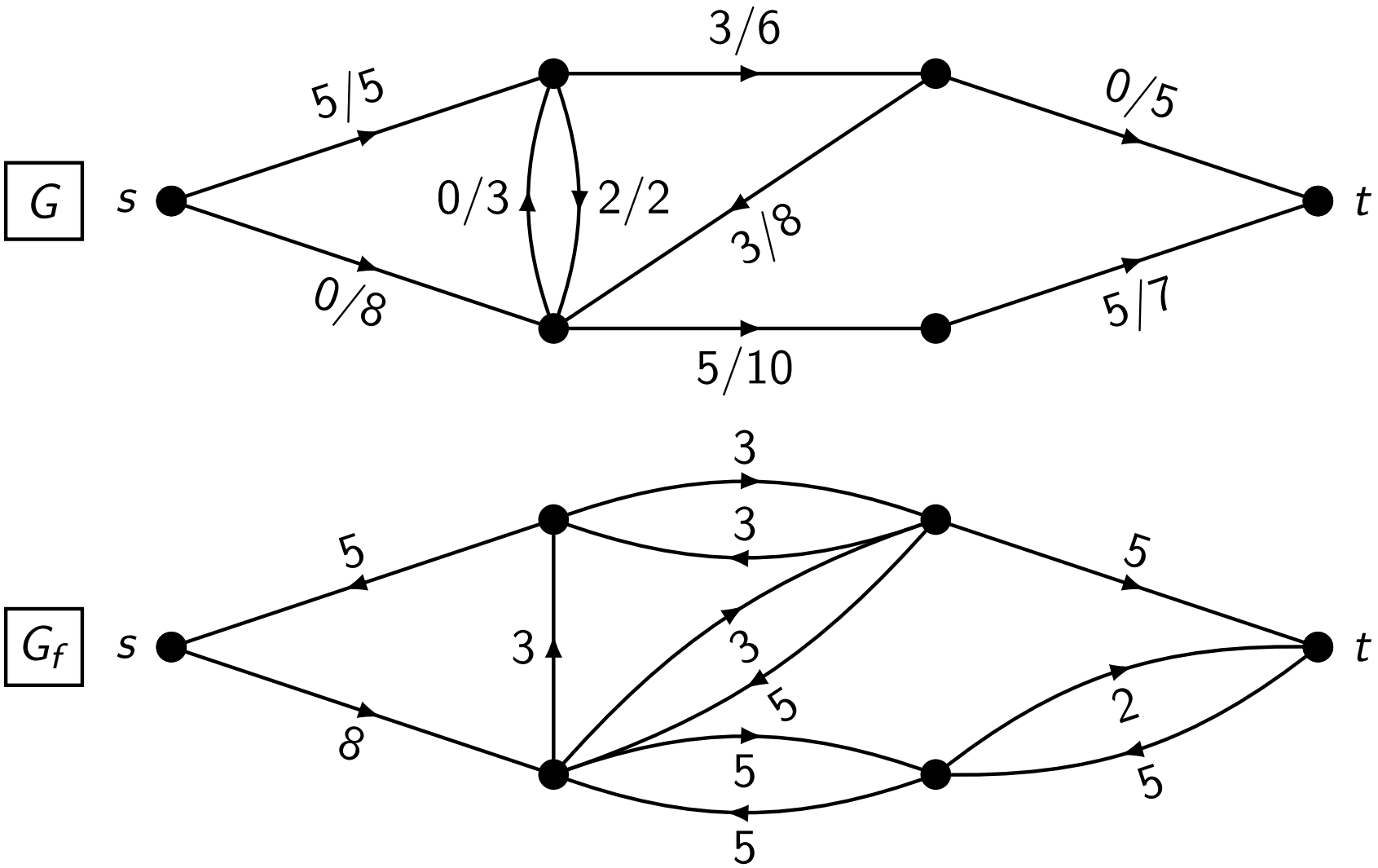

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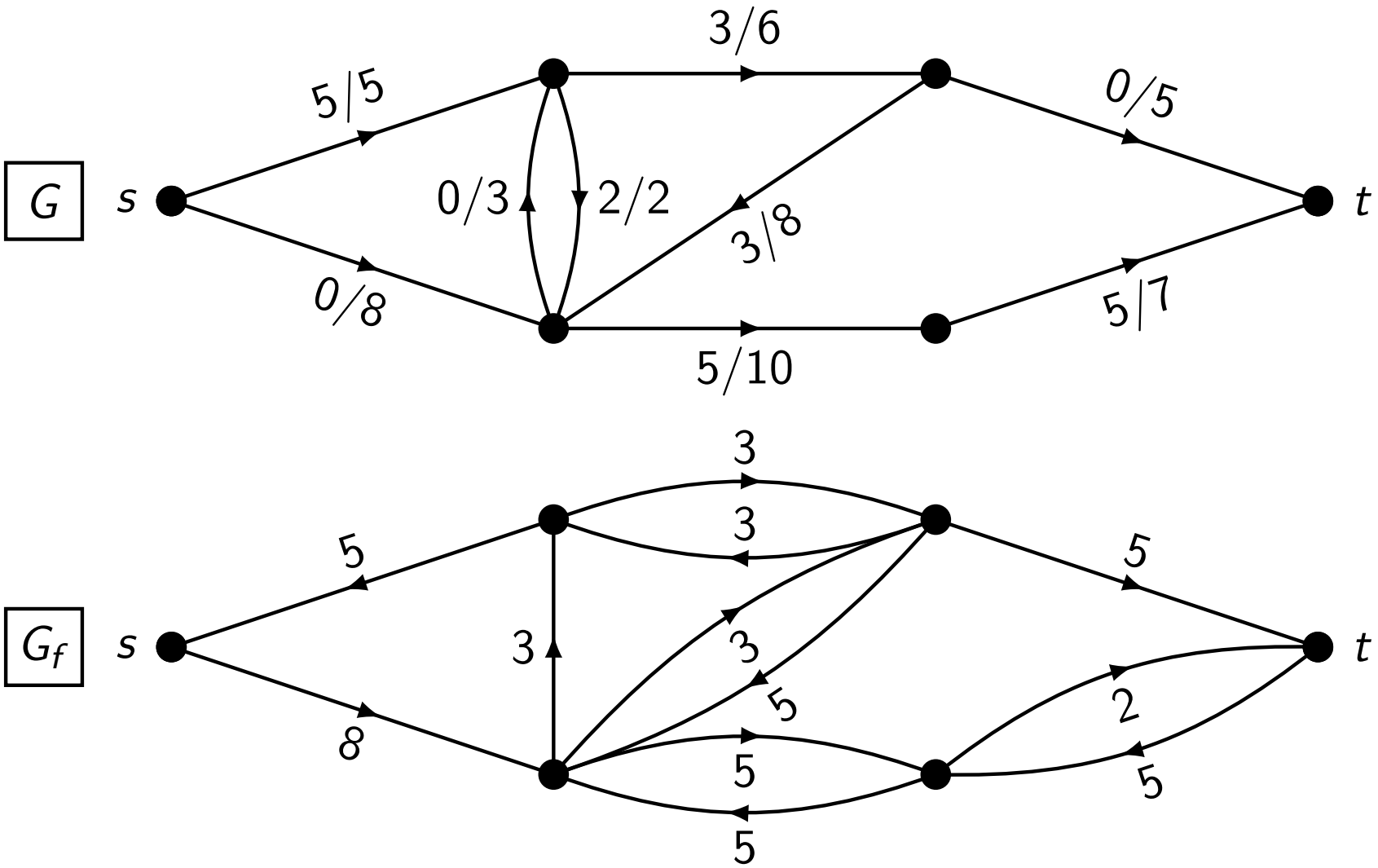

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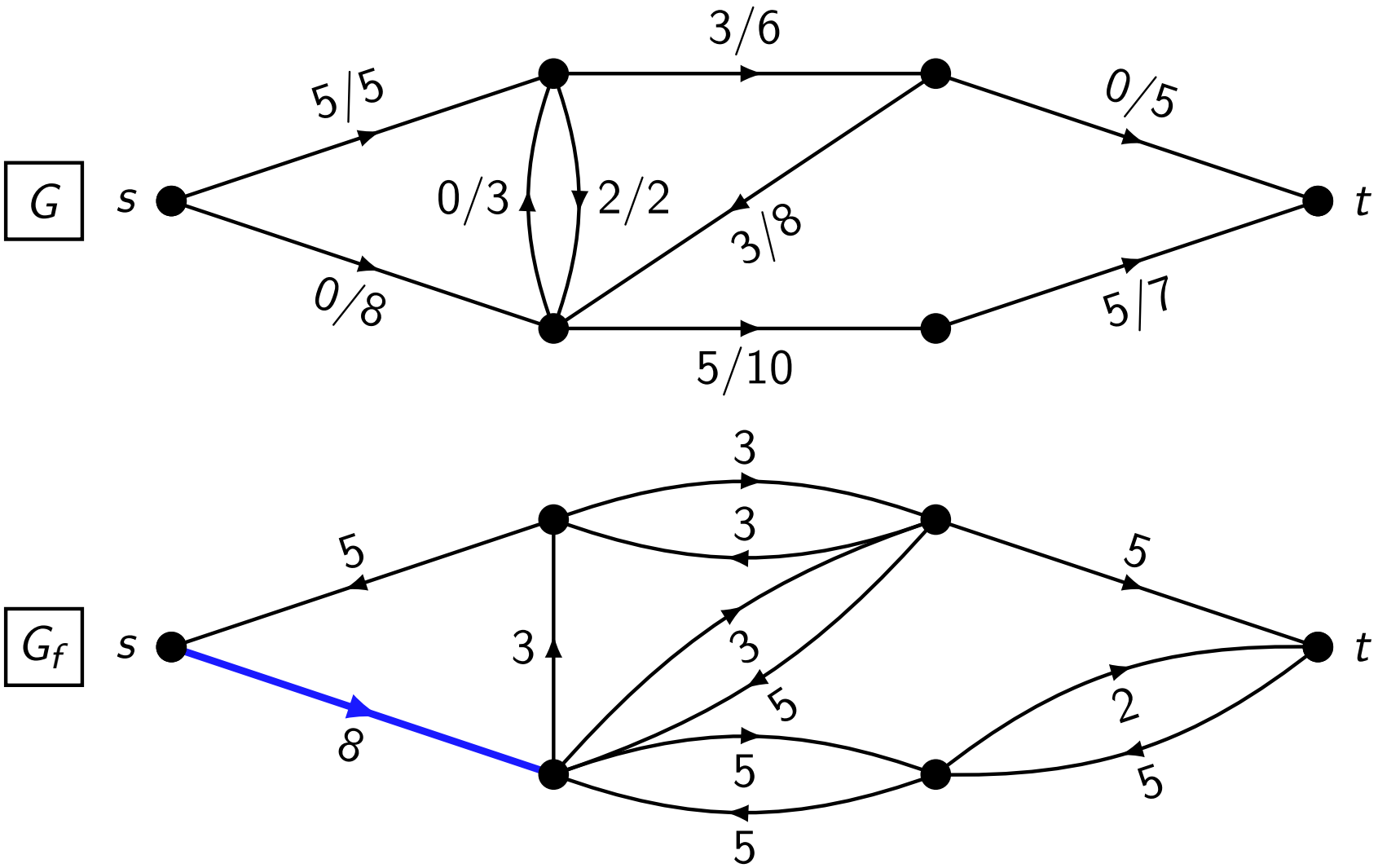

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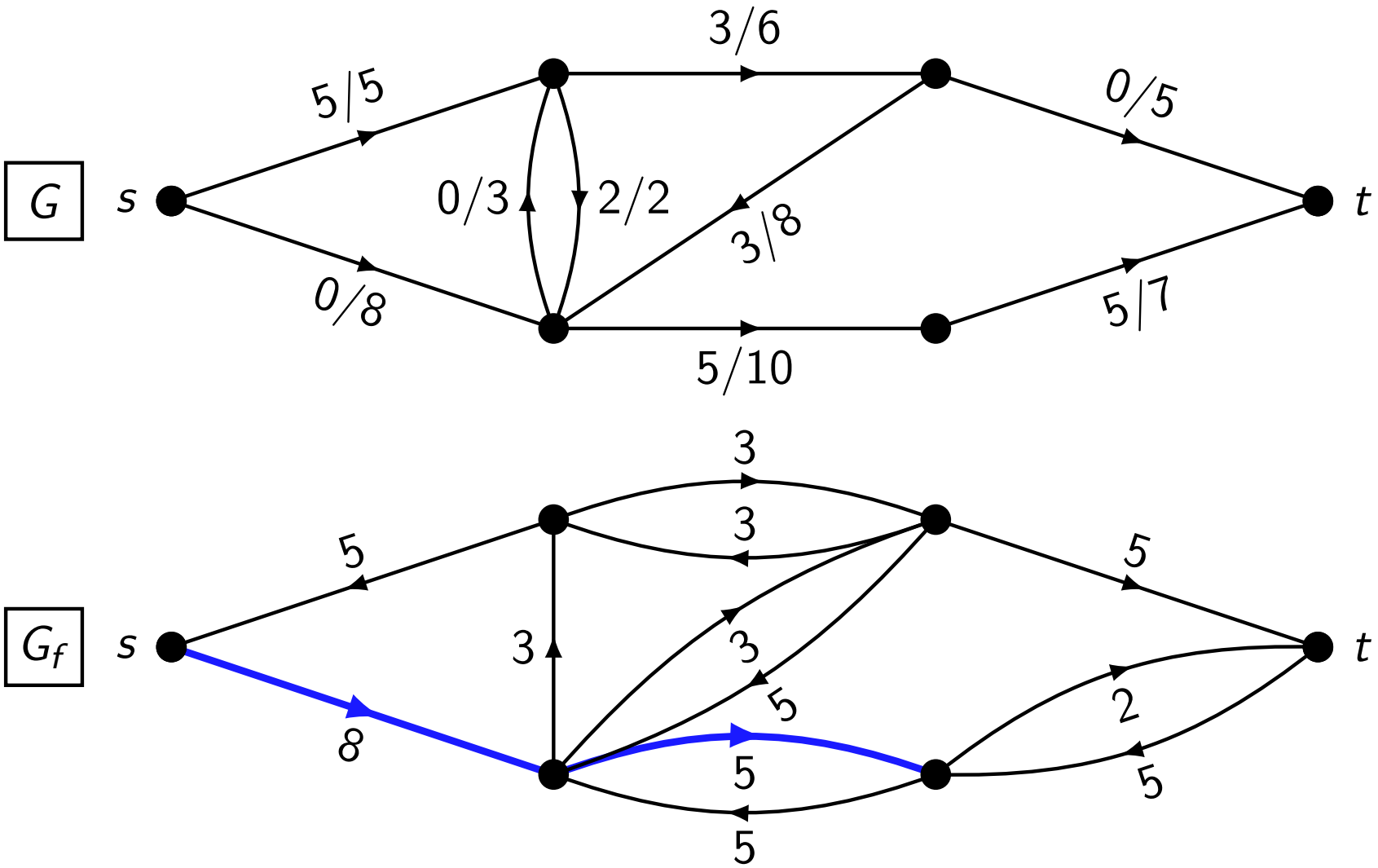

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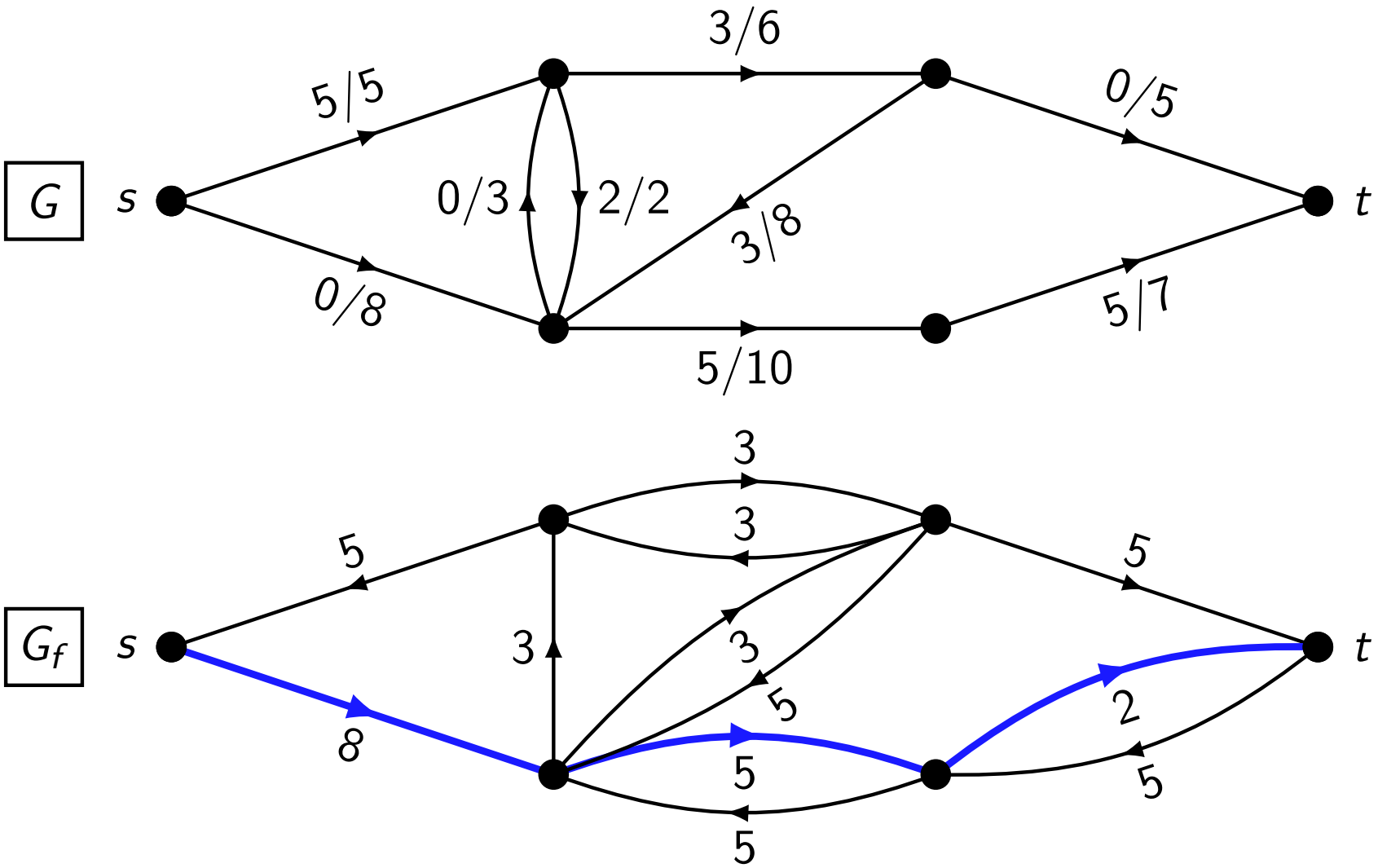

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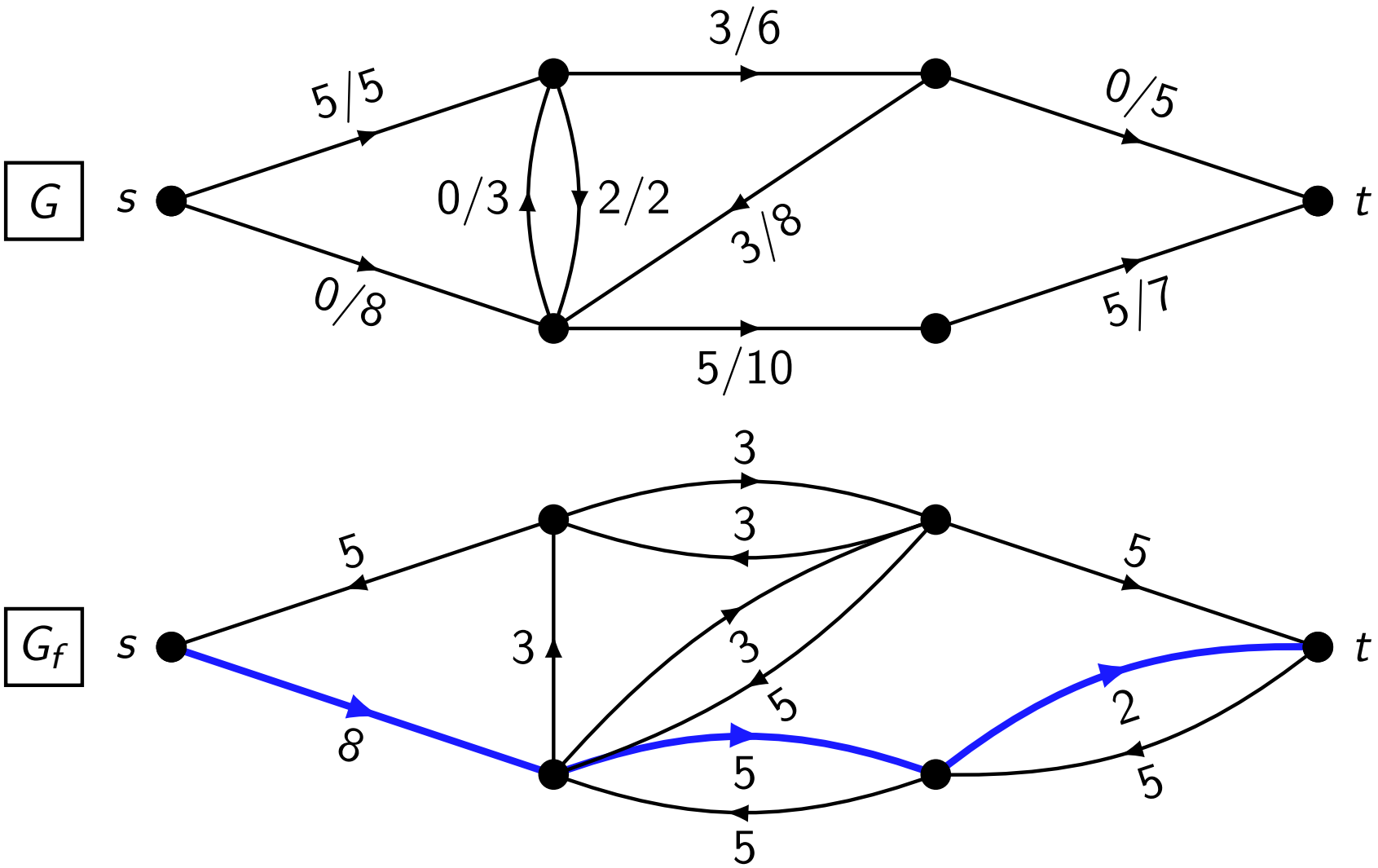

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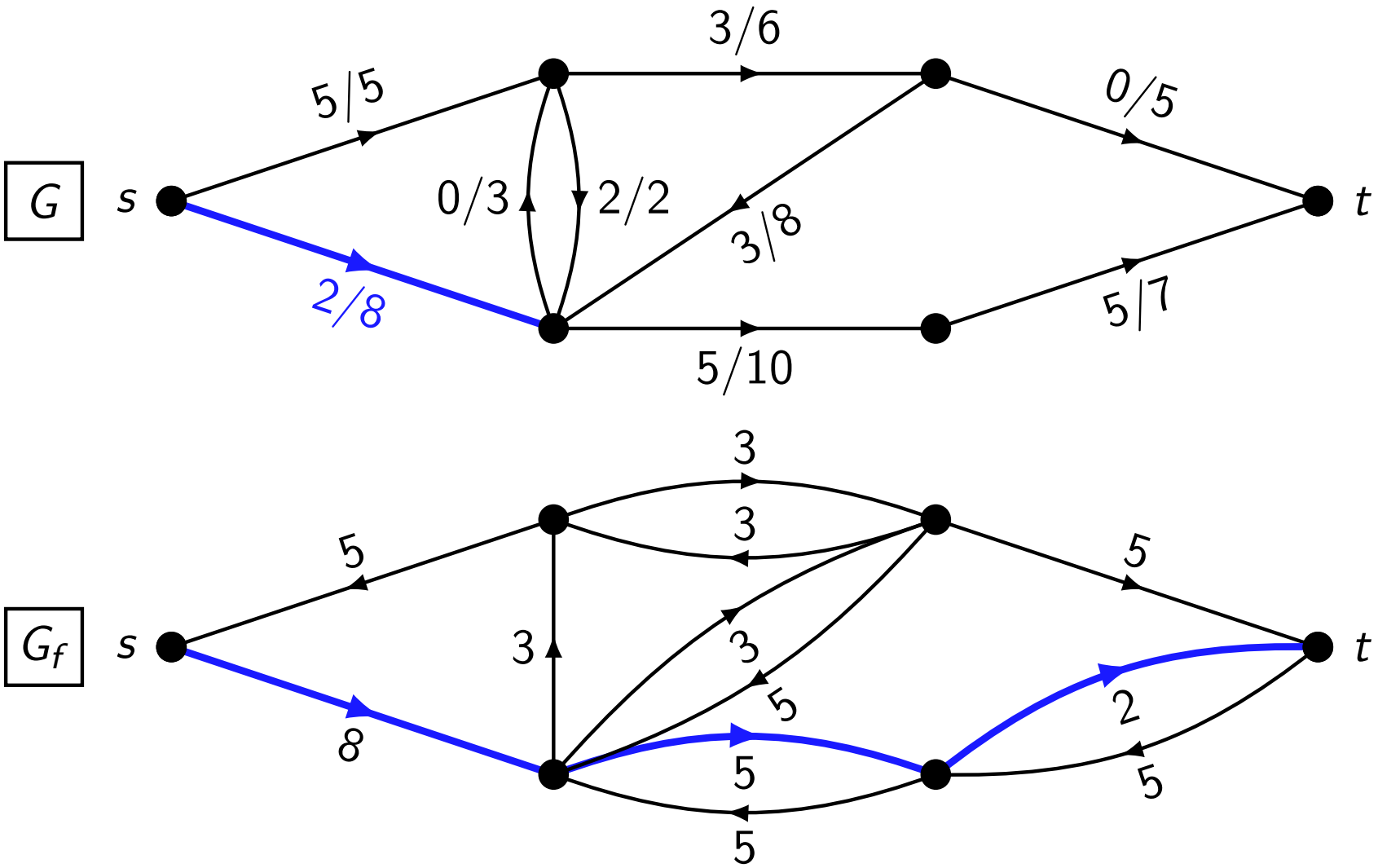

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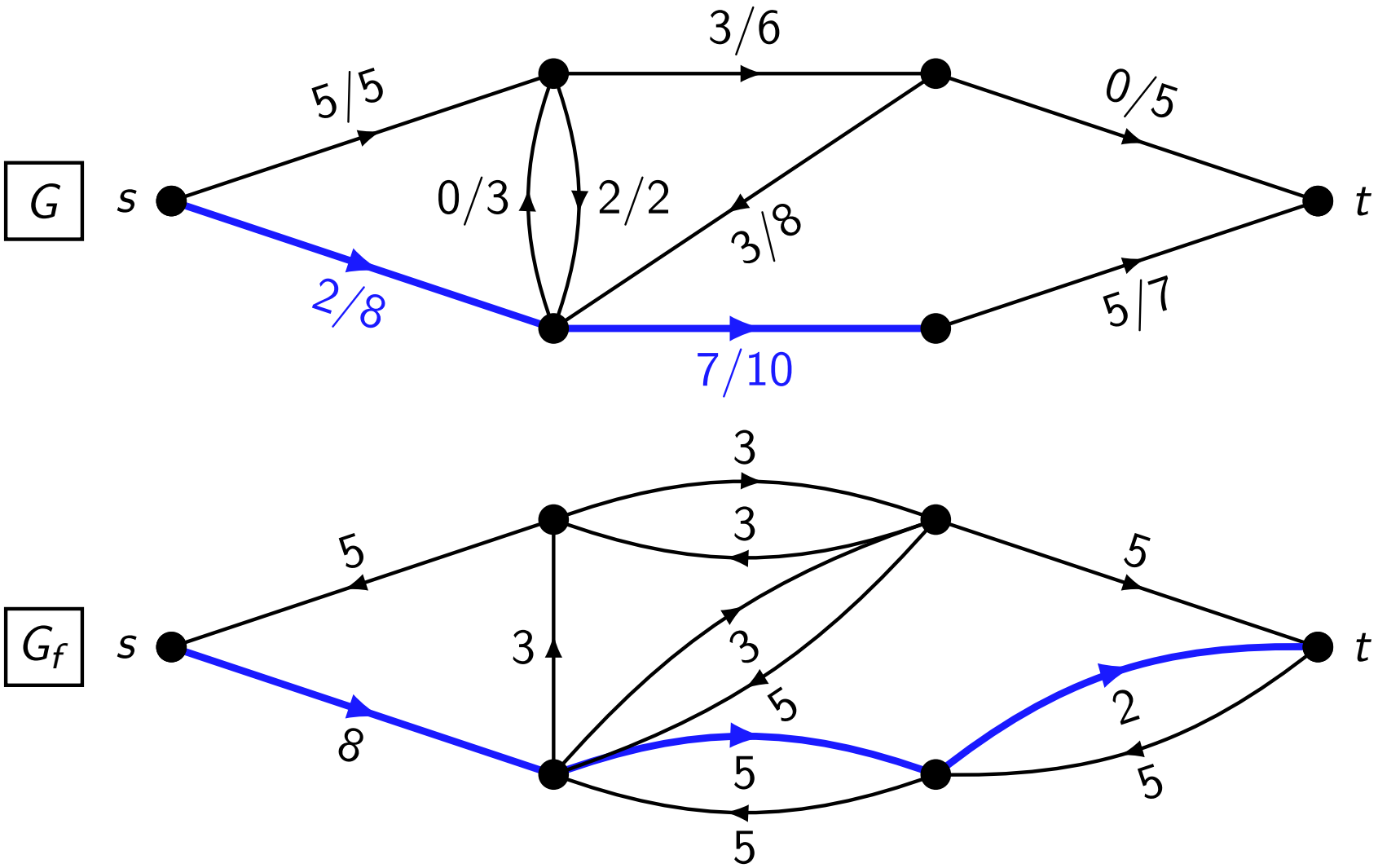

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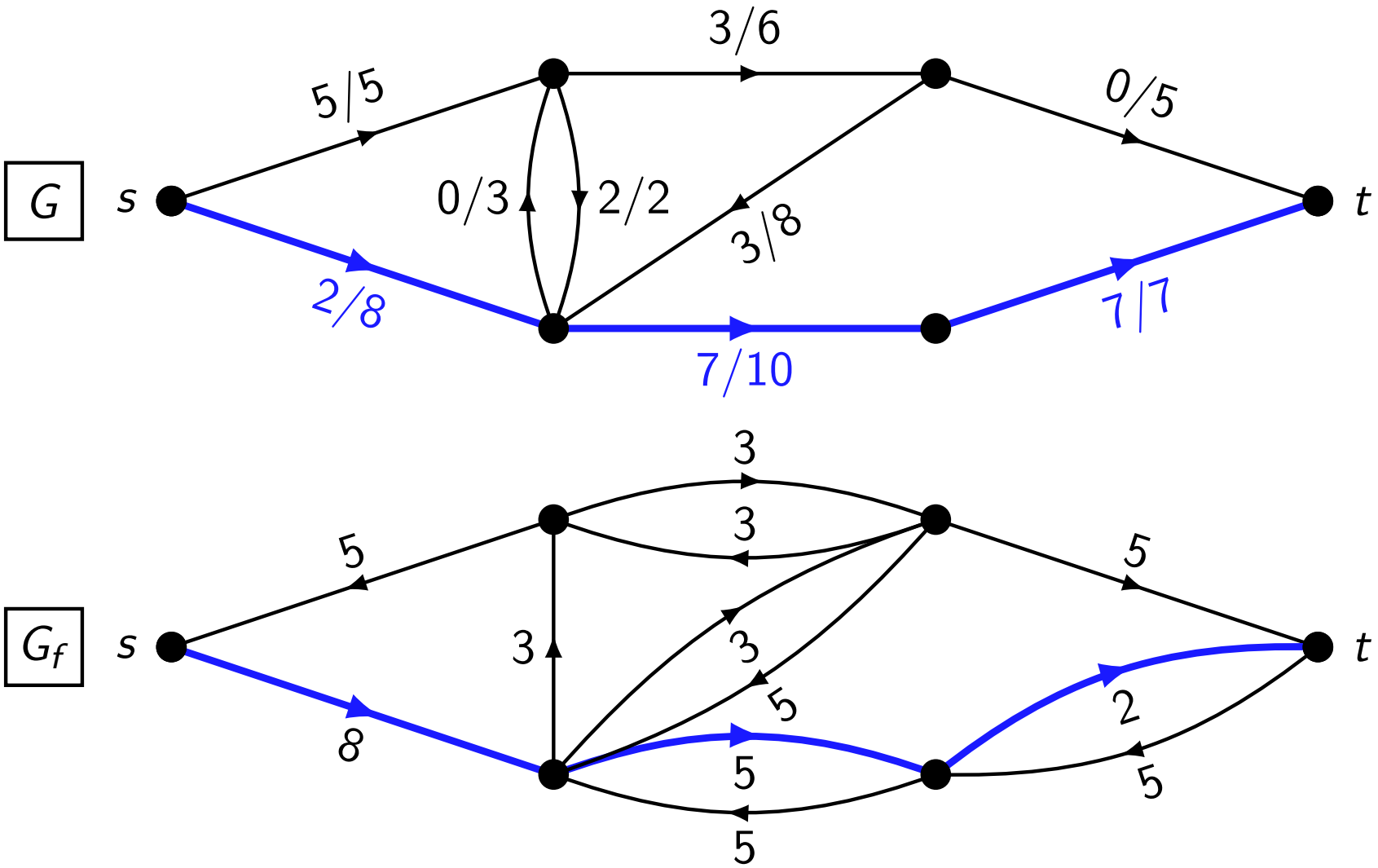

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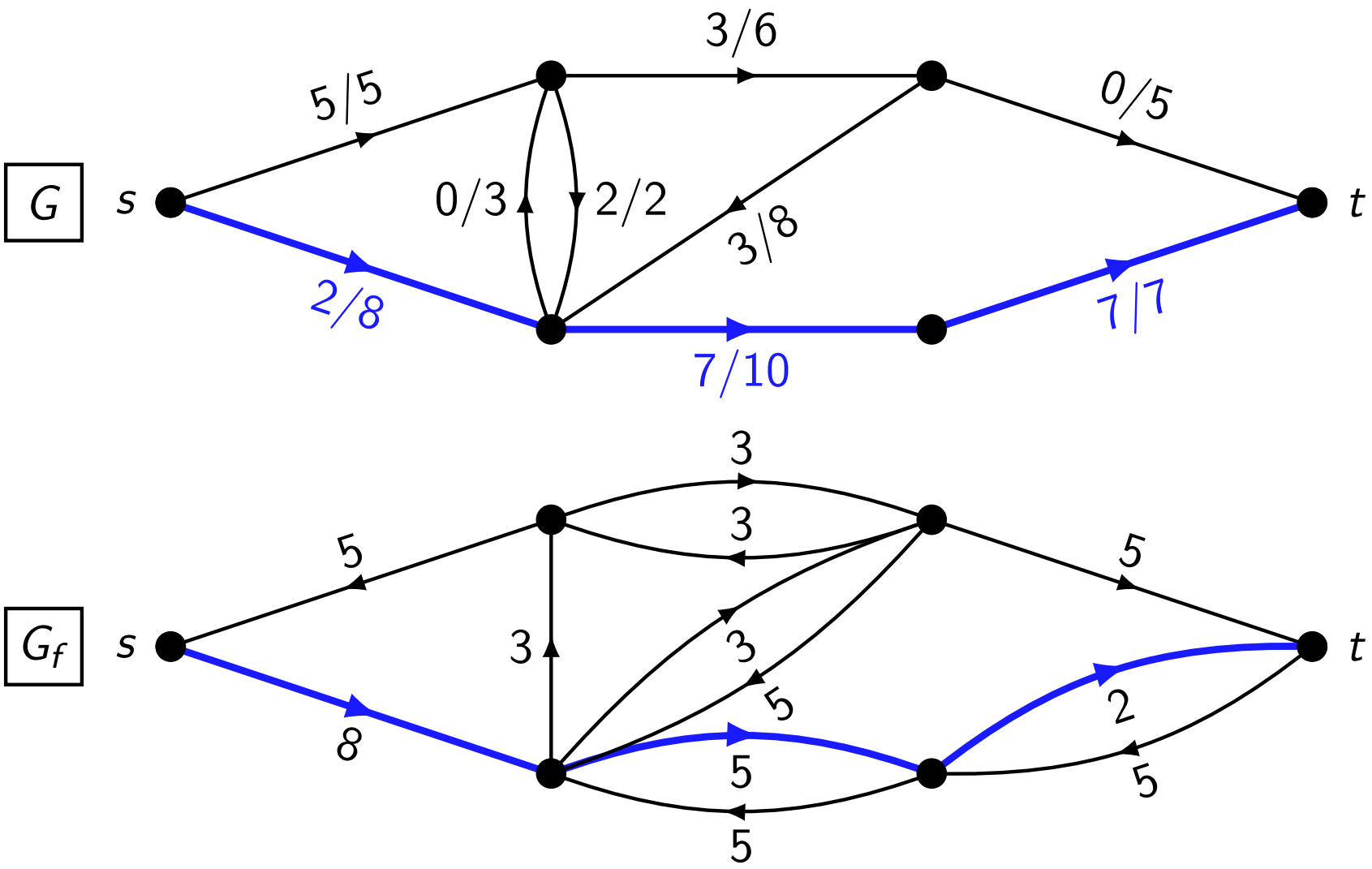
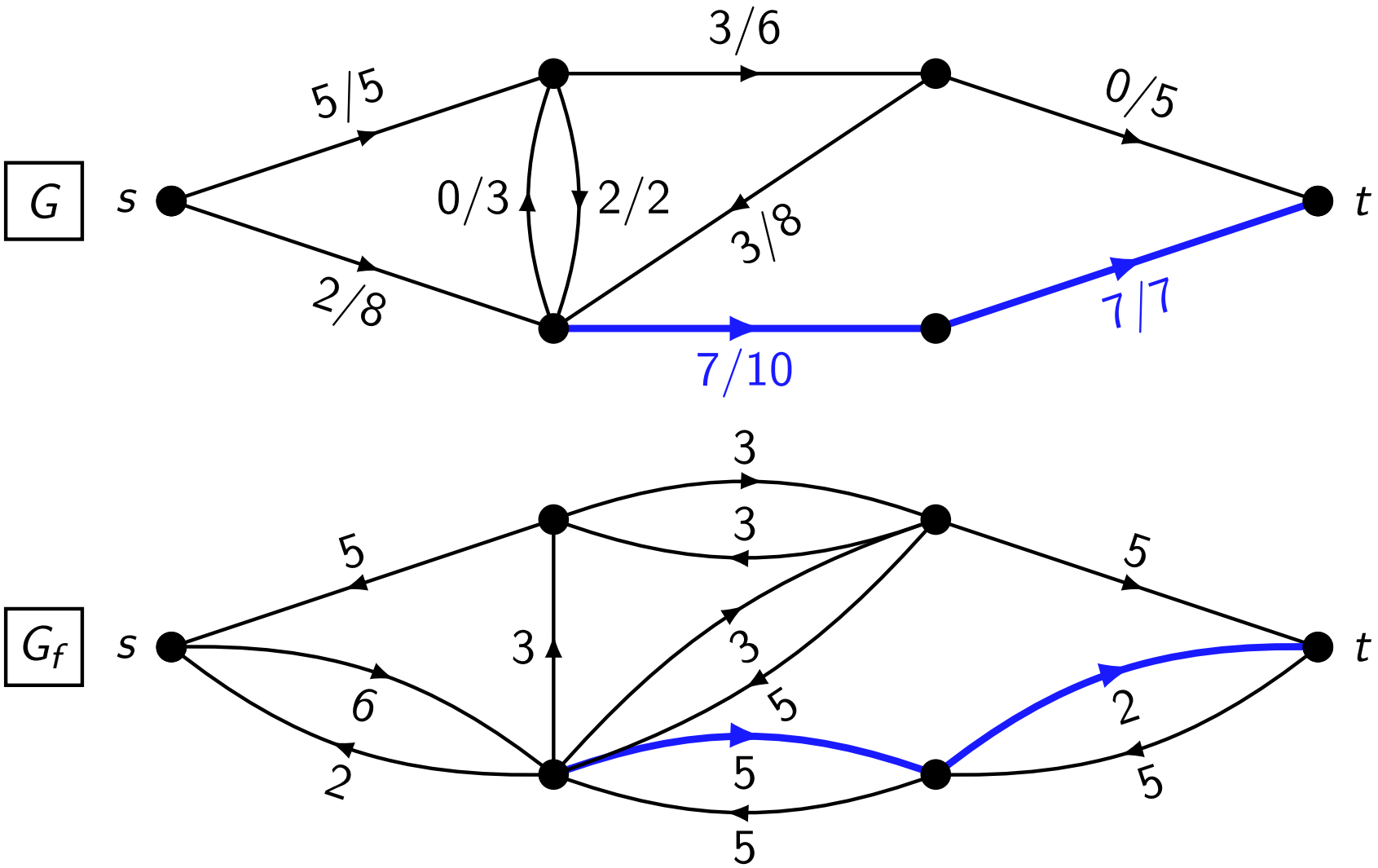


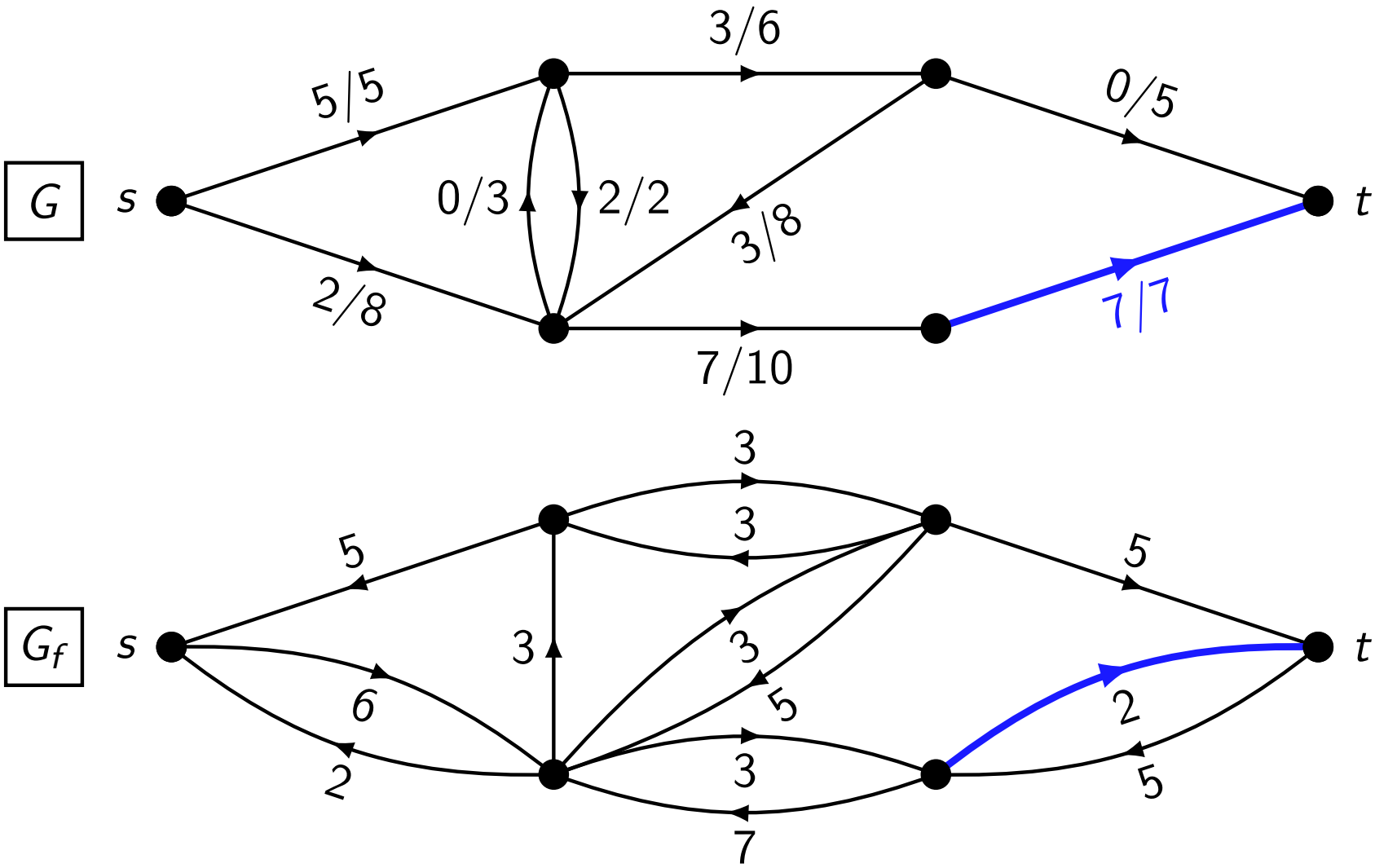
Update G_f along the path.

Worked example



Update G_f along the path.

Worked example



Update G_f along the path.

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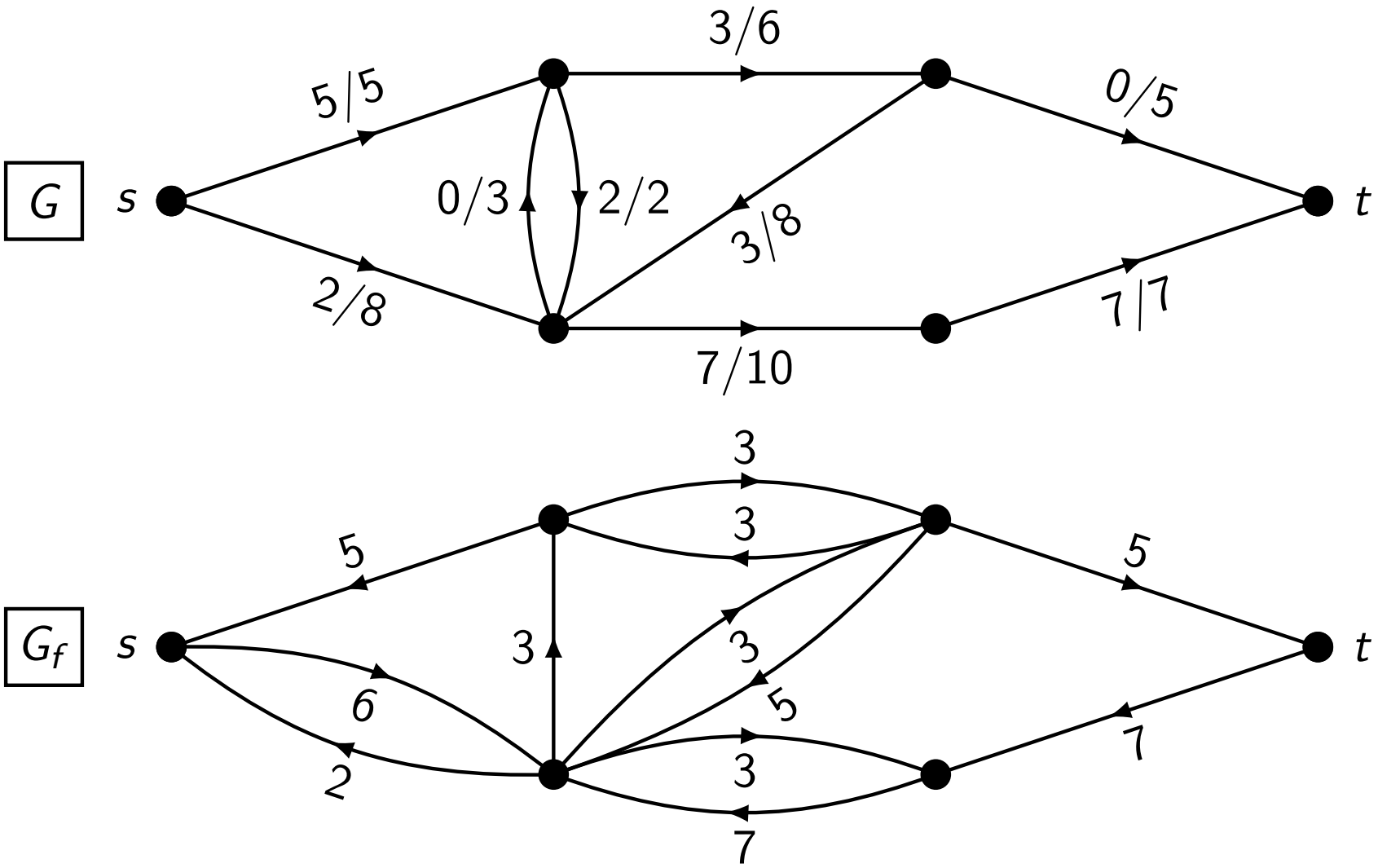
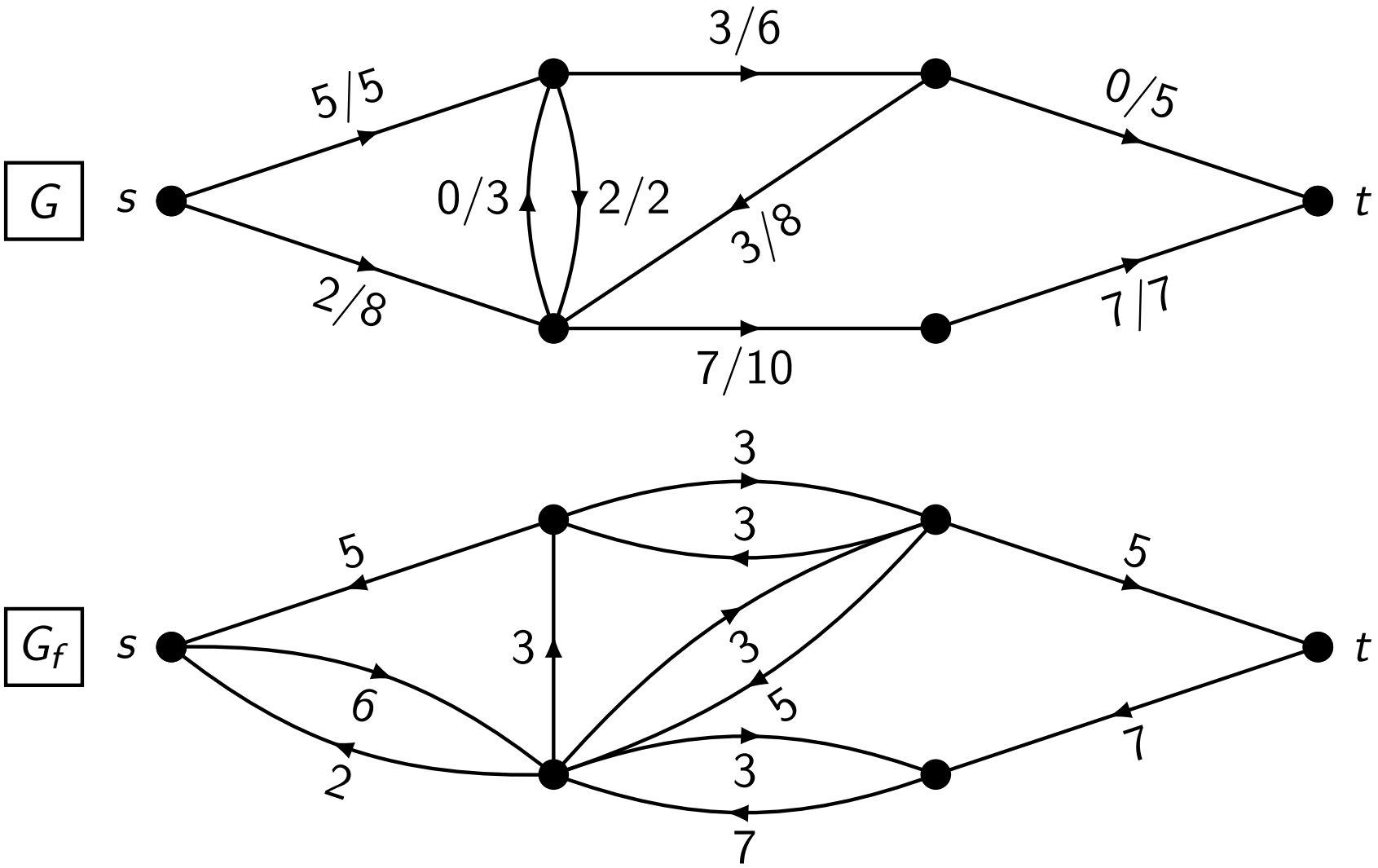


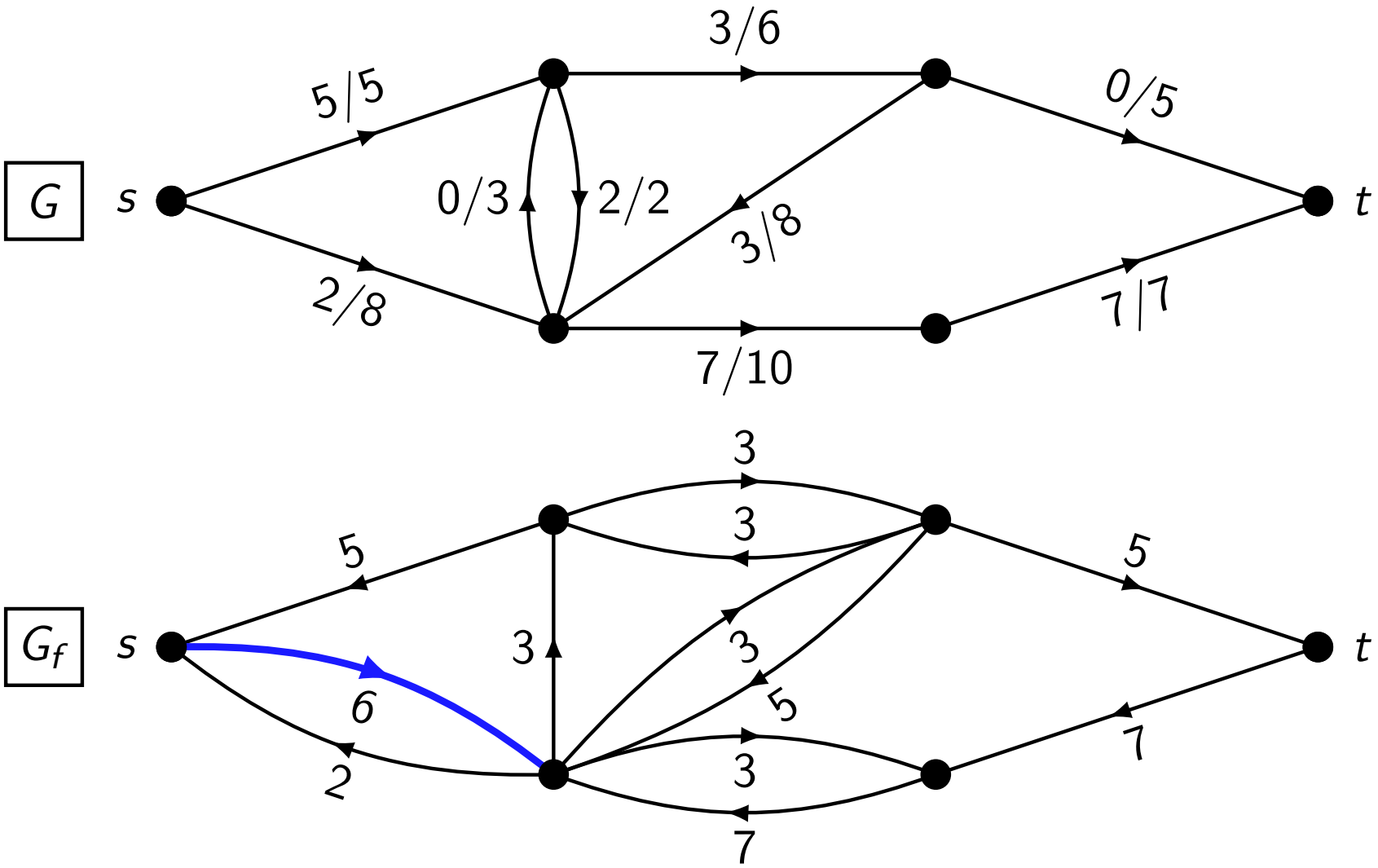
Update G_f along the path.

Worked example



Apply depth-first search to find an augmenting path in G_f.

Worked example



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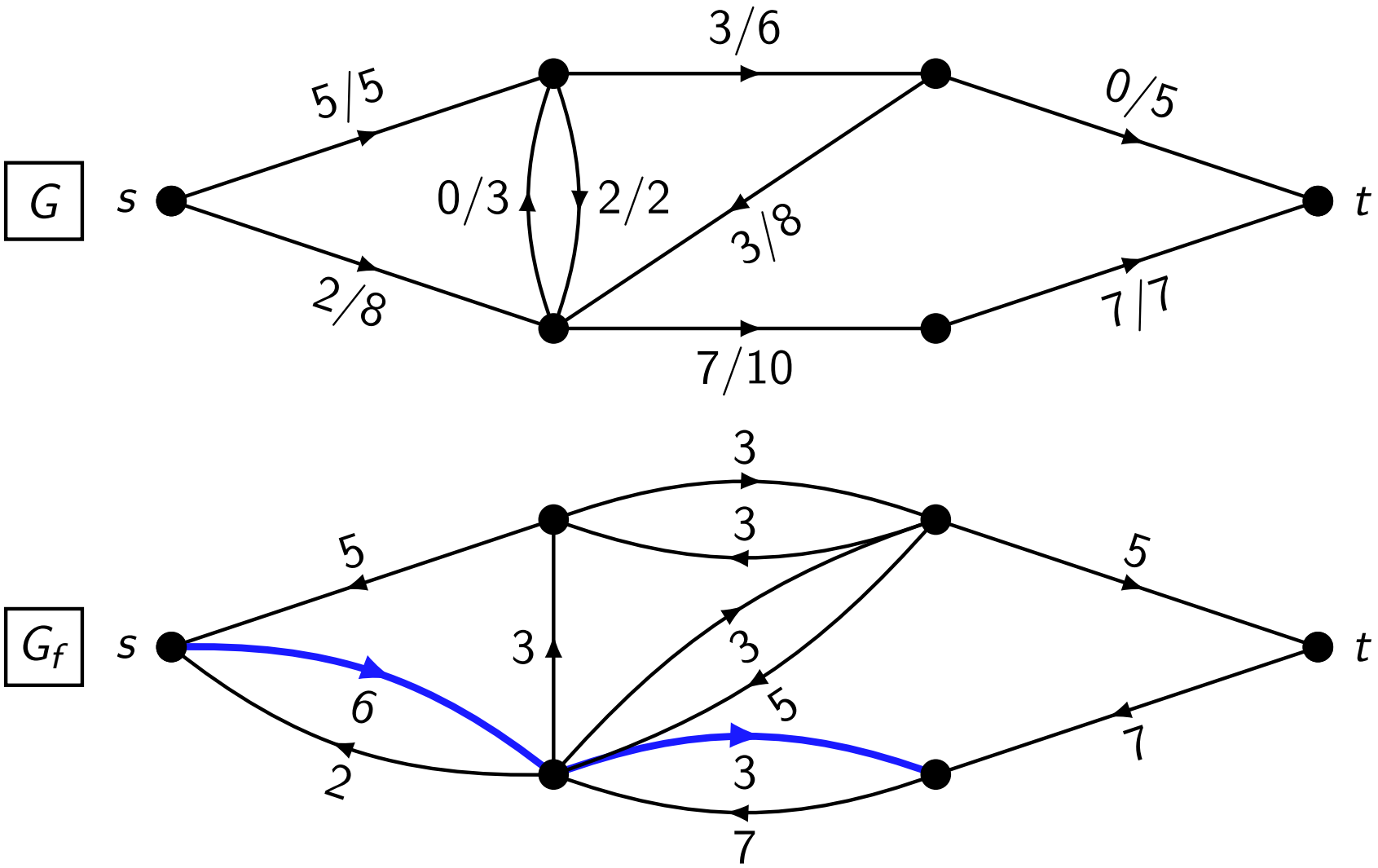

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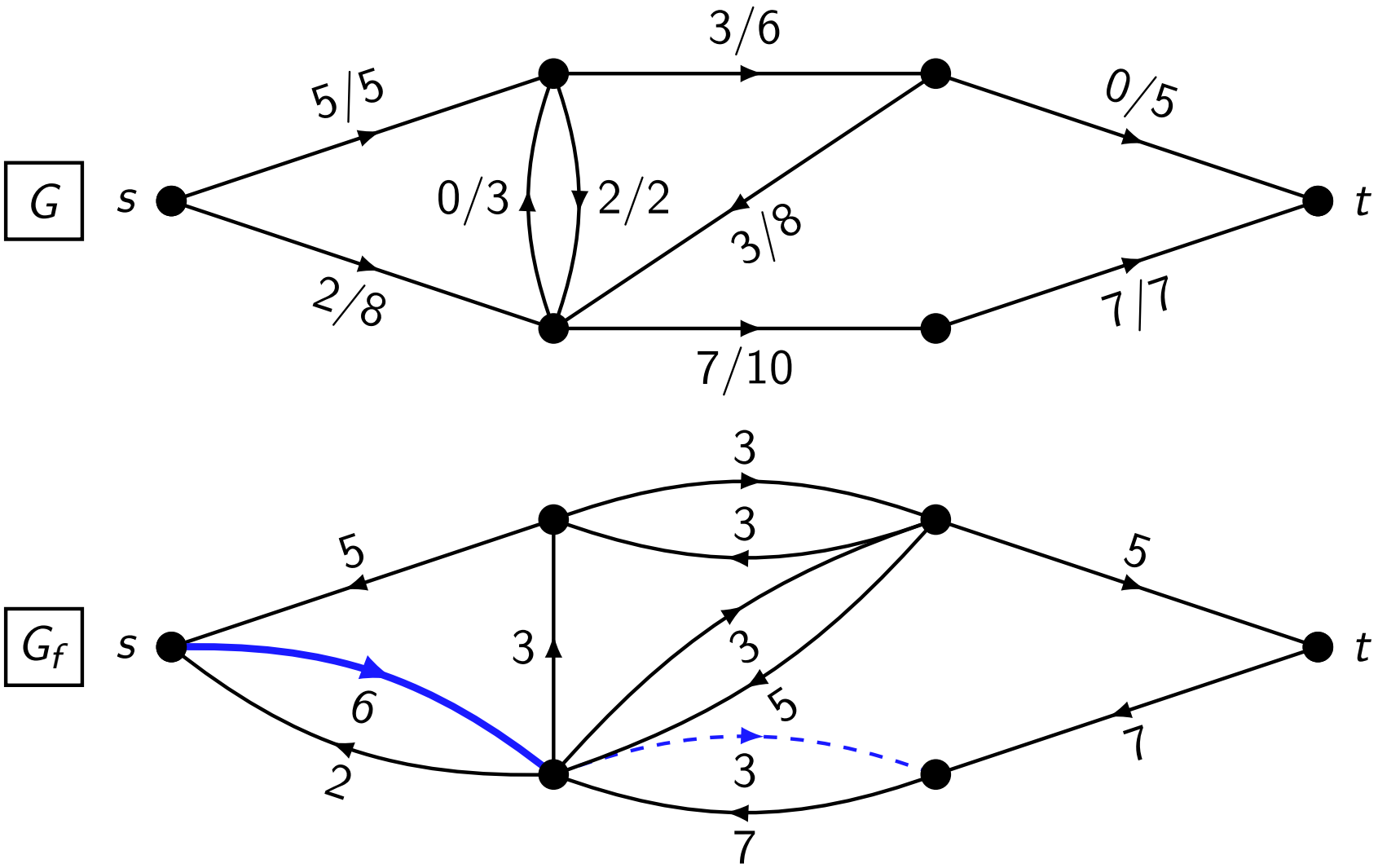

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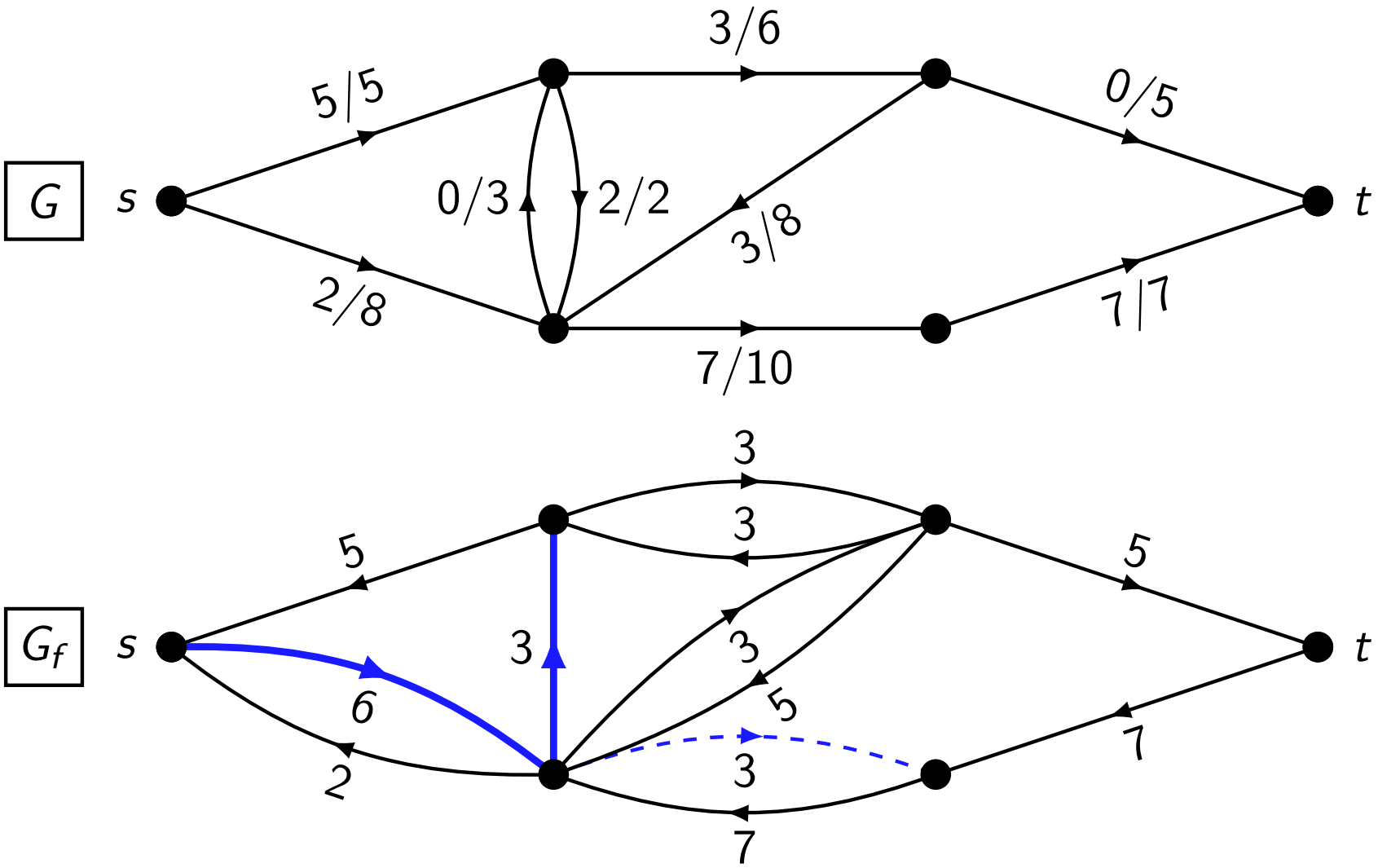

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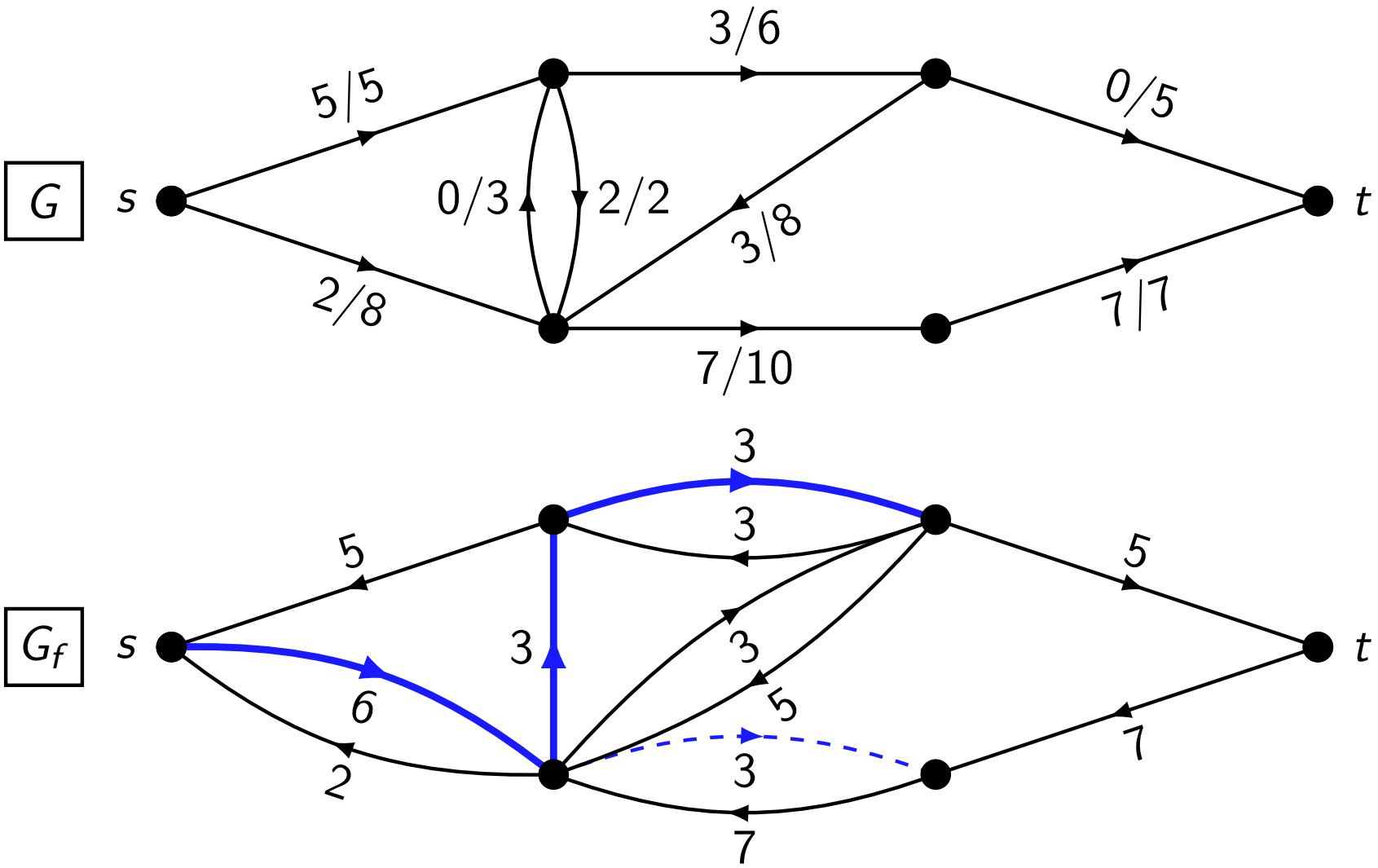

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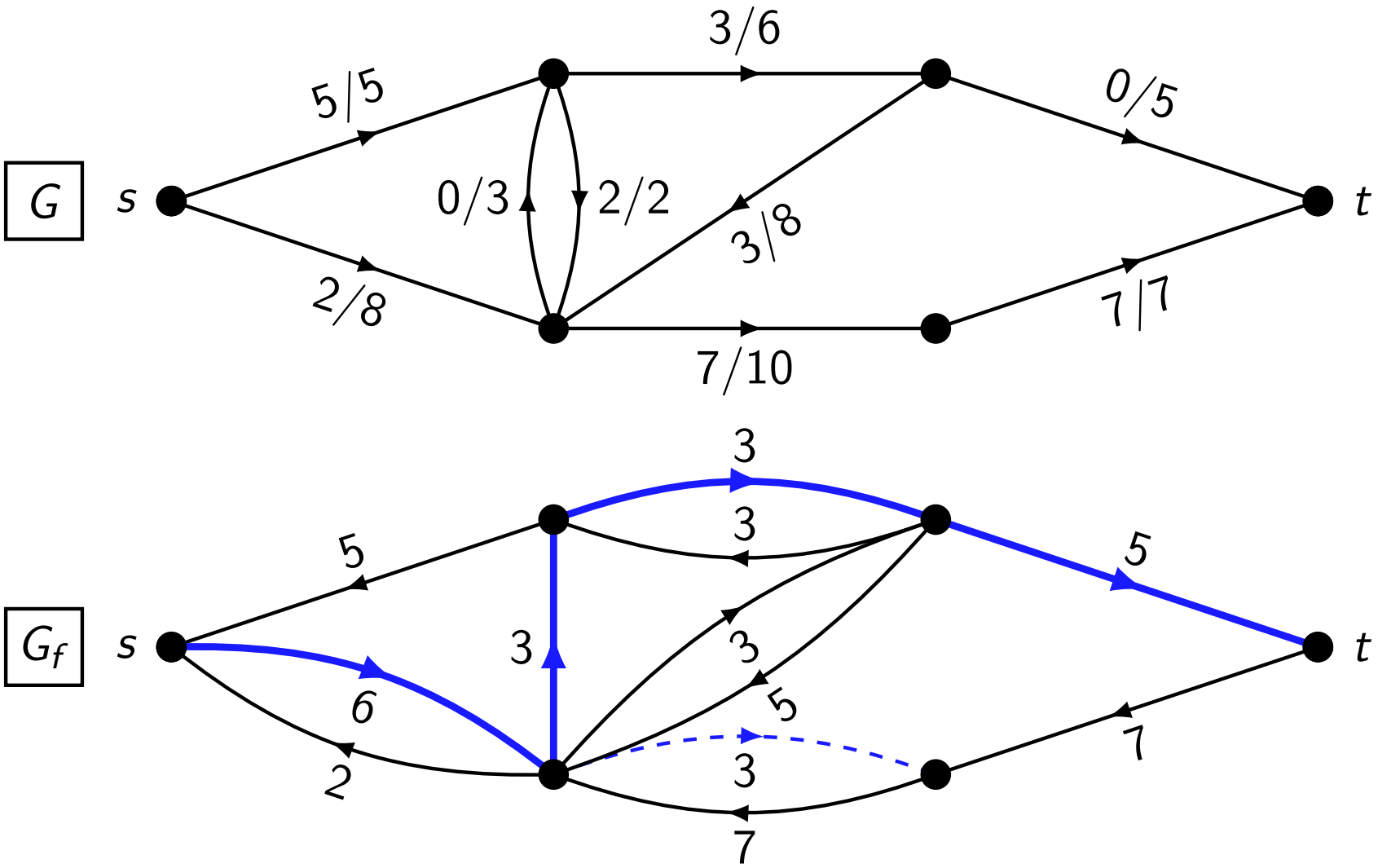

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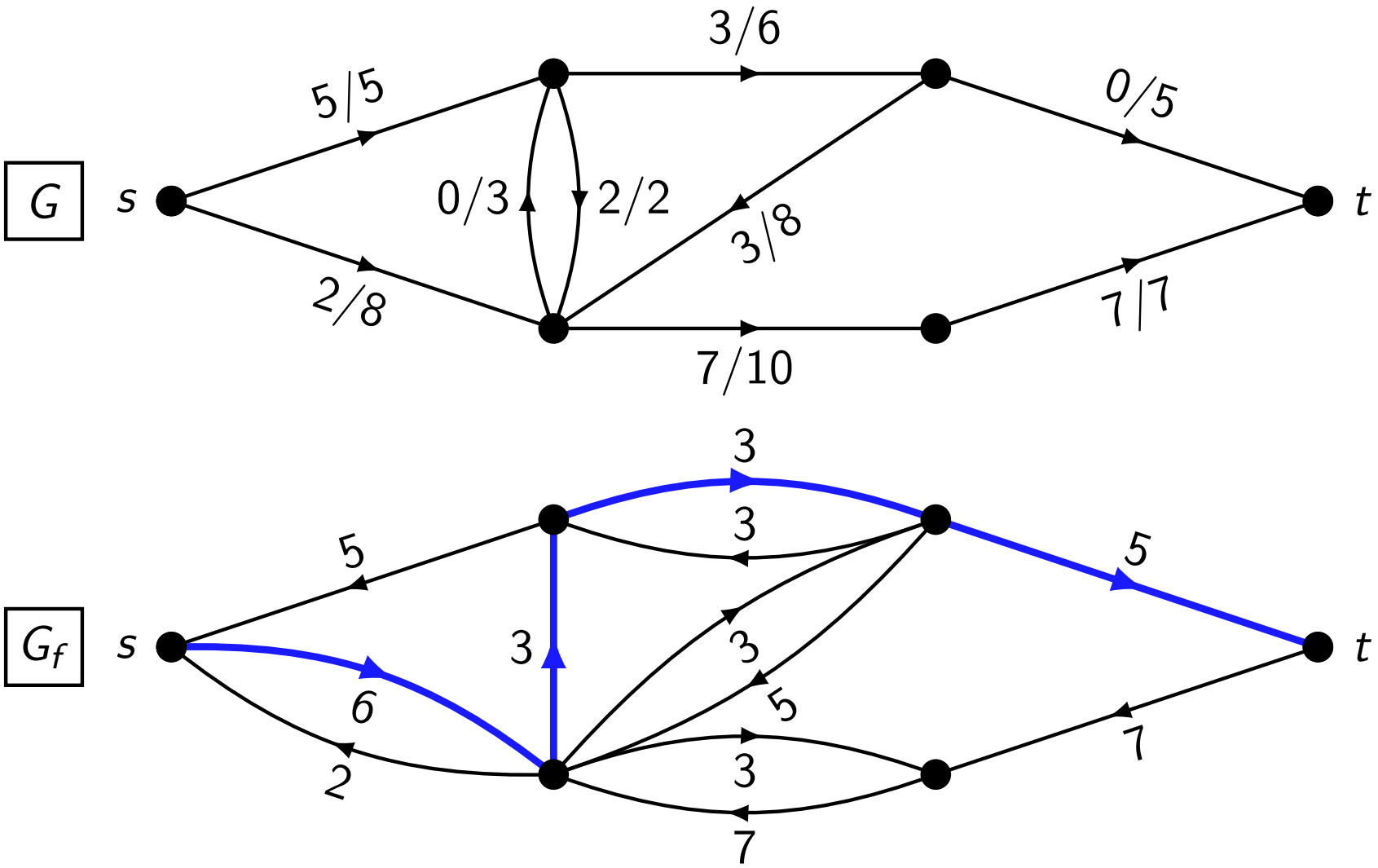

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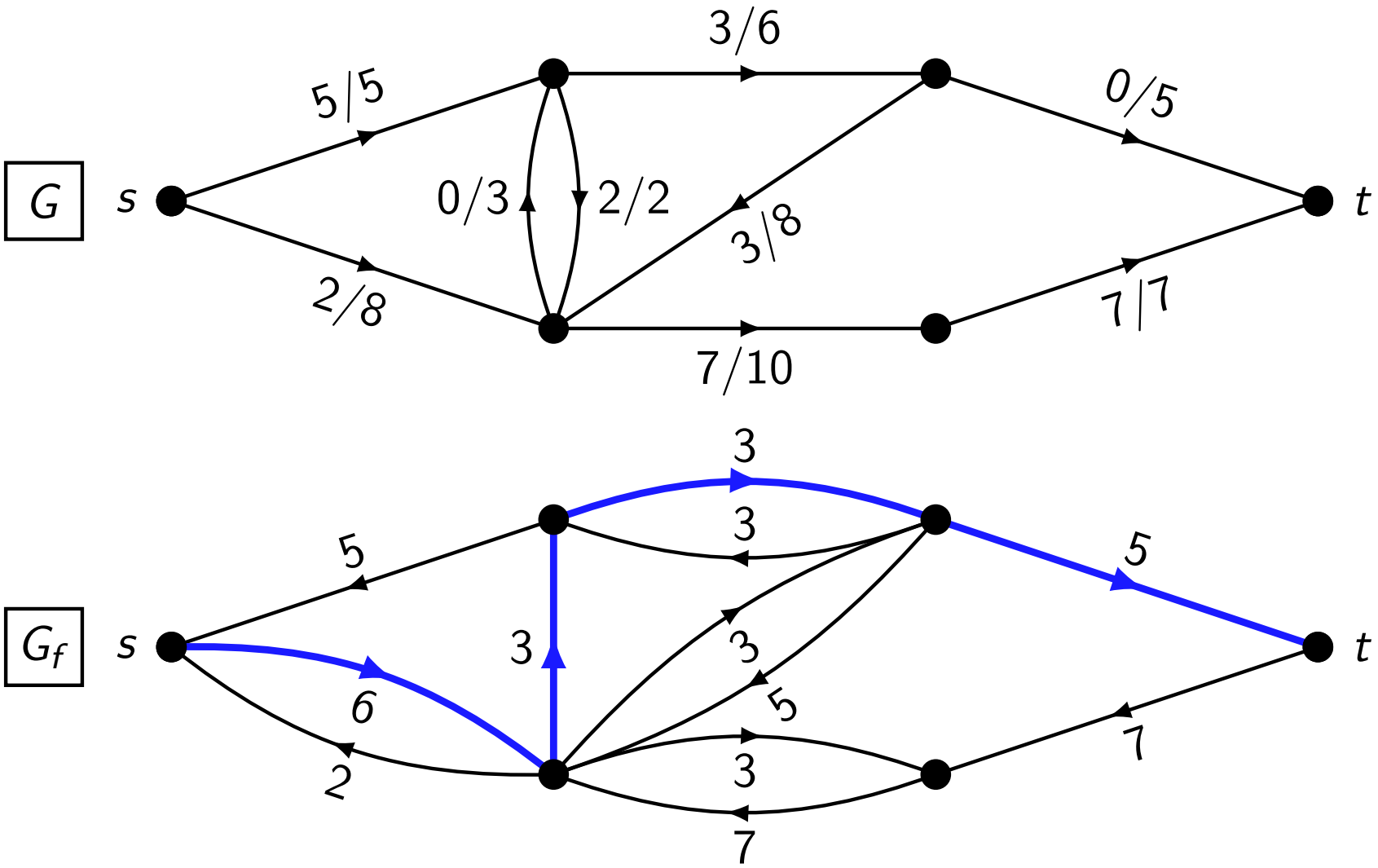
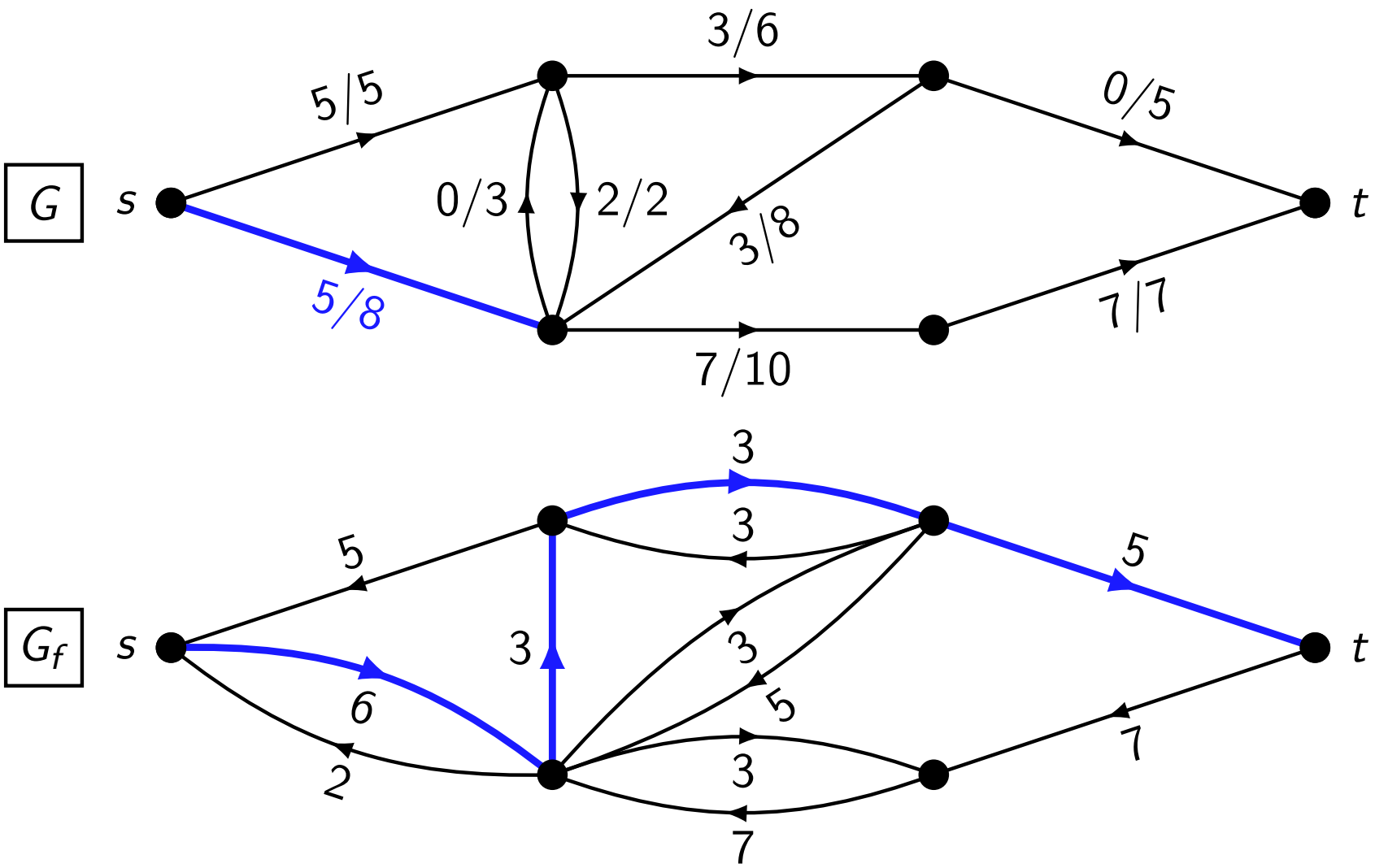


Push flow along the path. (This path has residual capacity 3.)

Worked example



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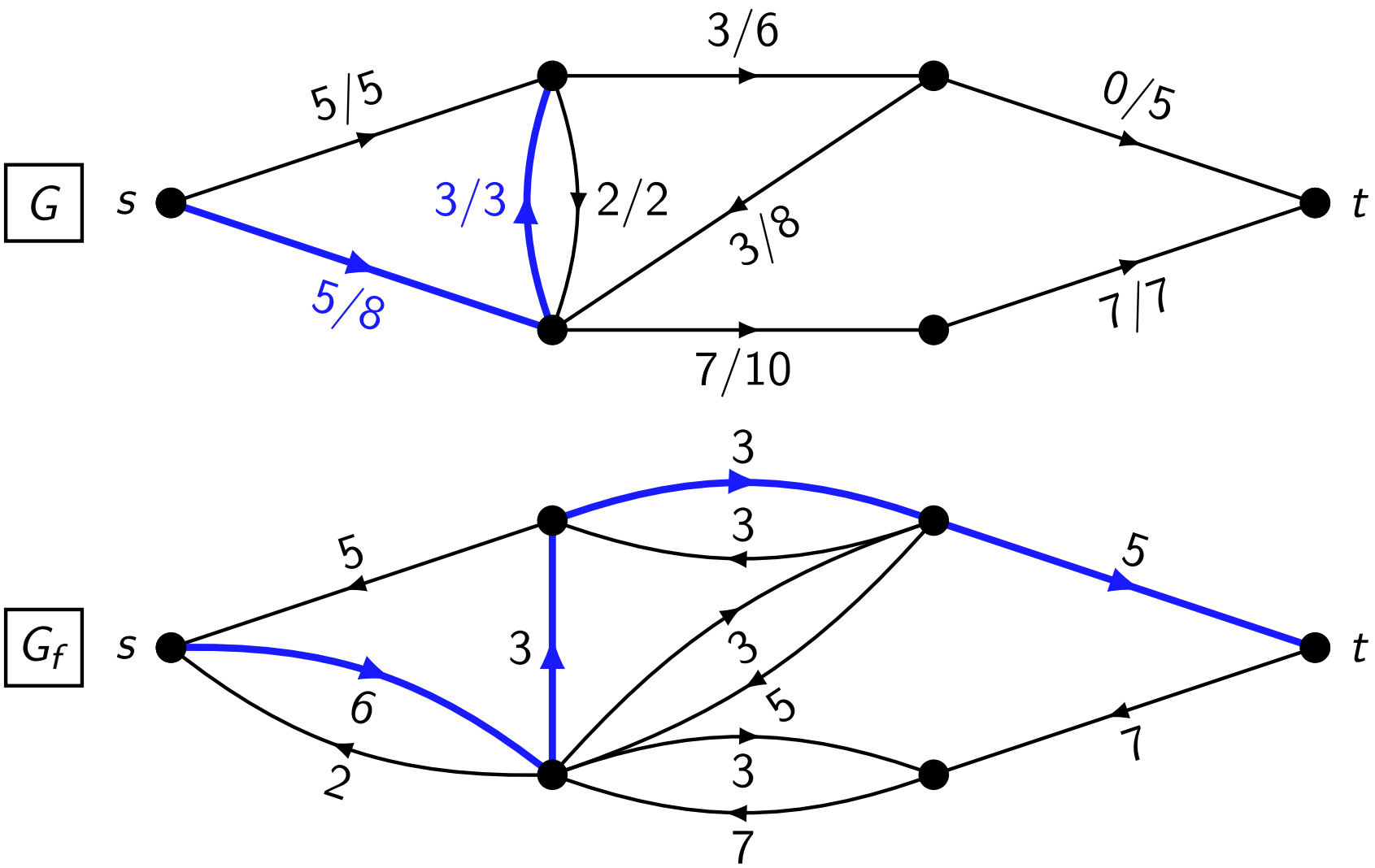

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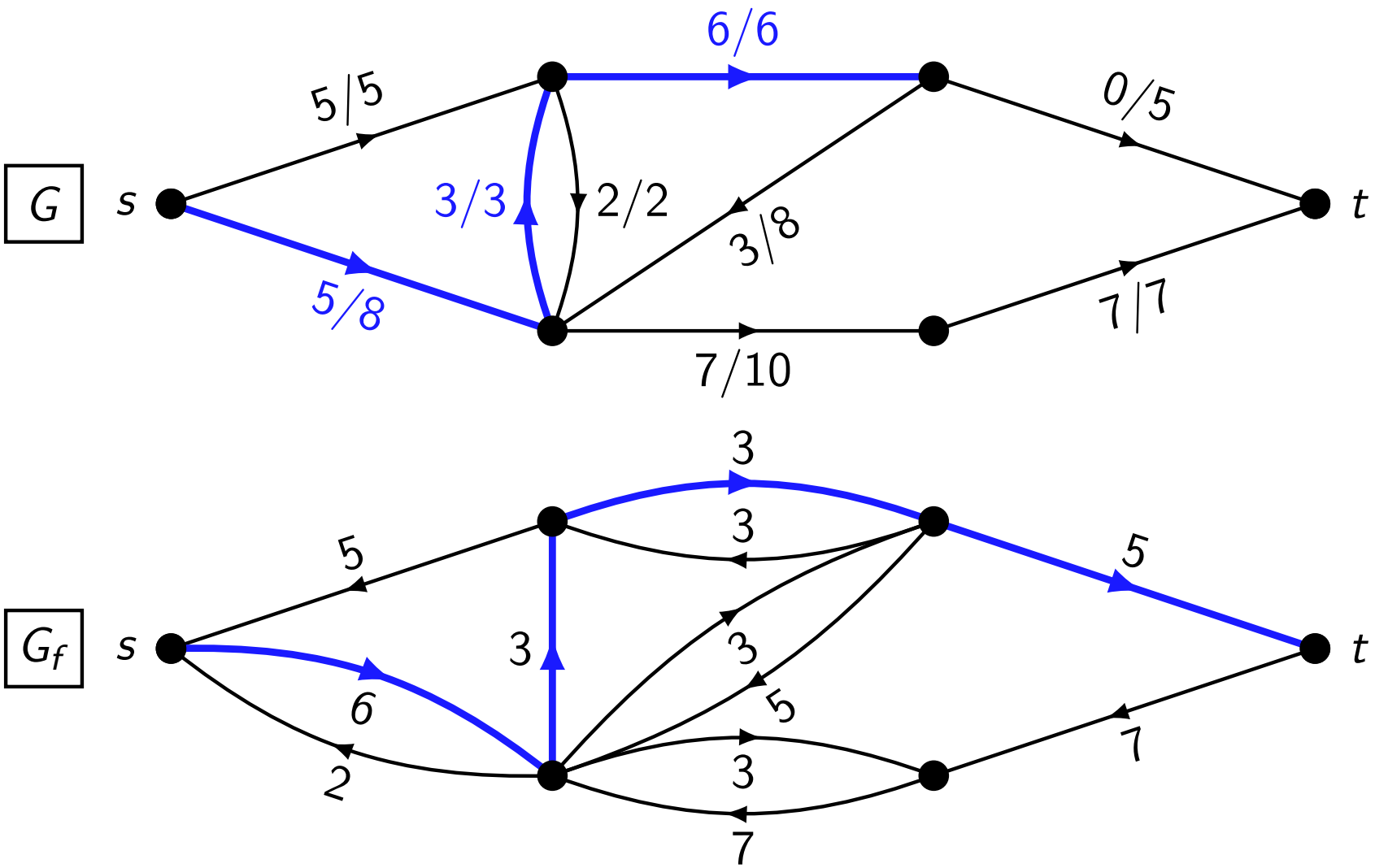

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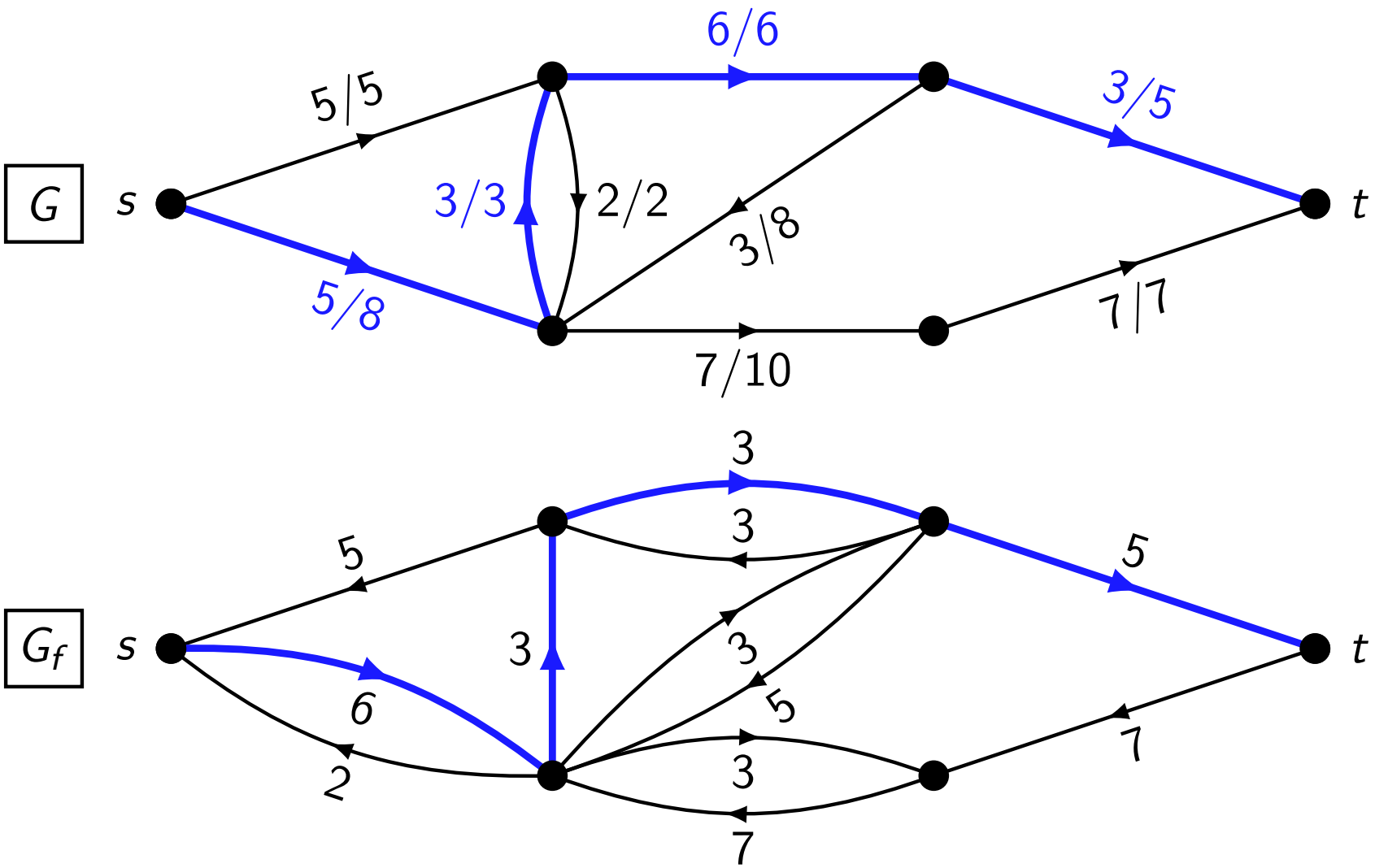

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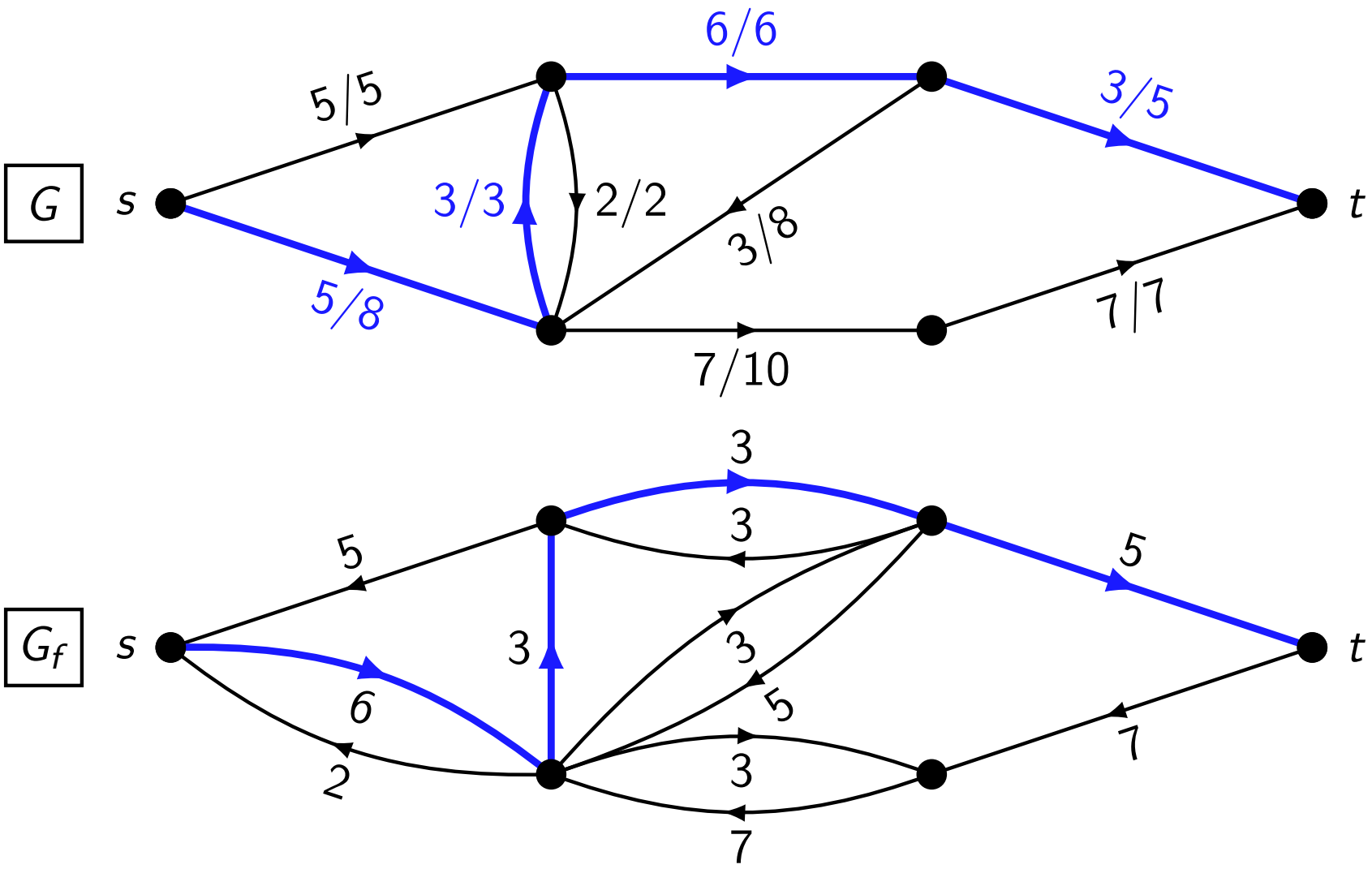
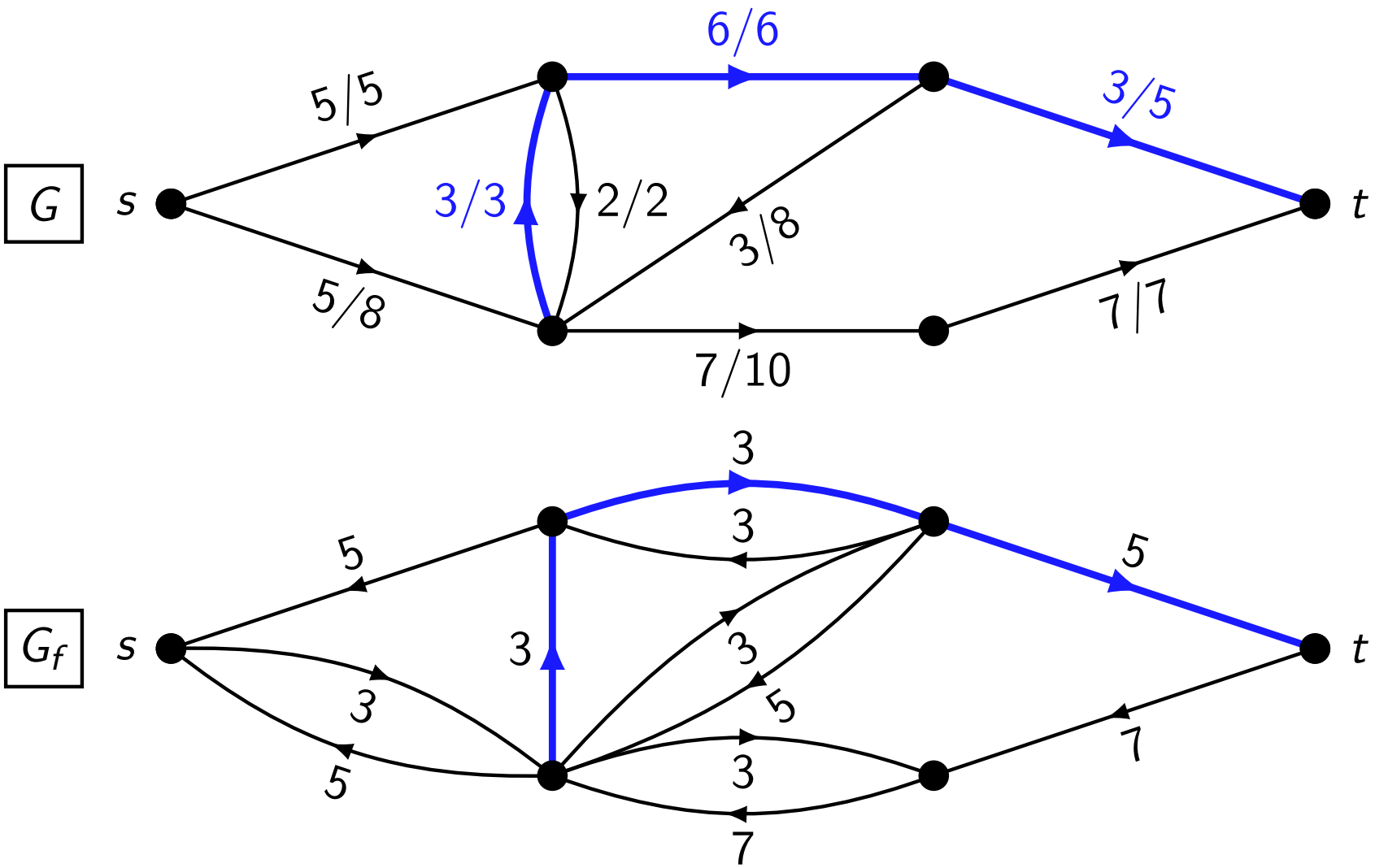


Update G_f along the path.

Worked example



Update G_f along the path.

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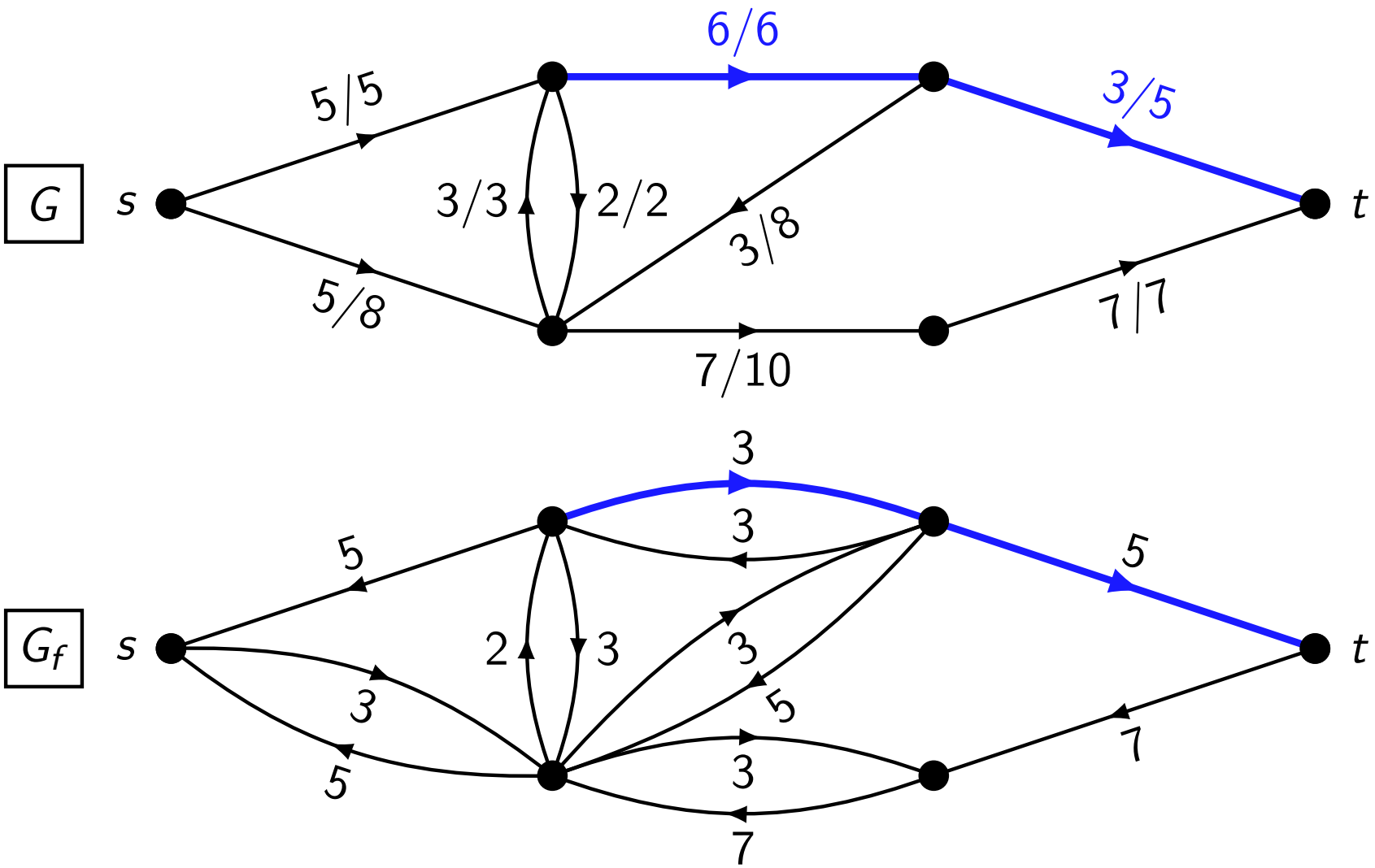

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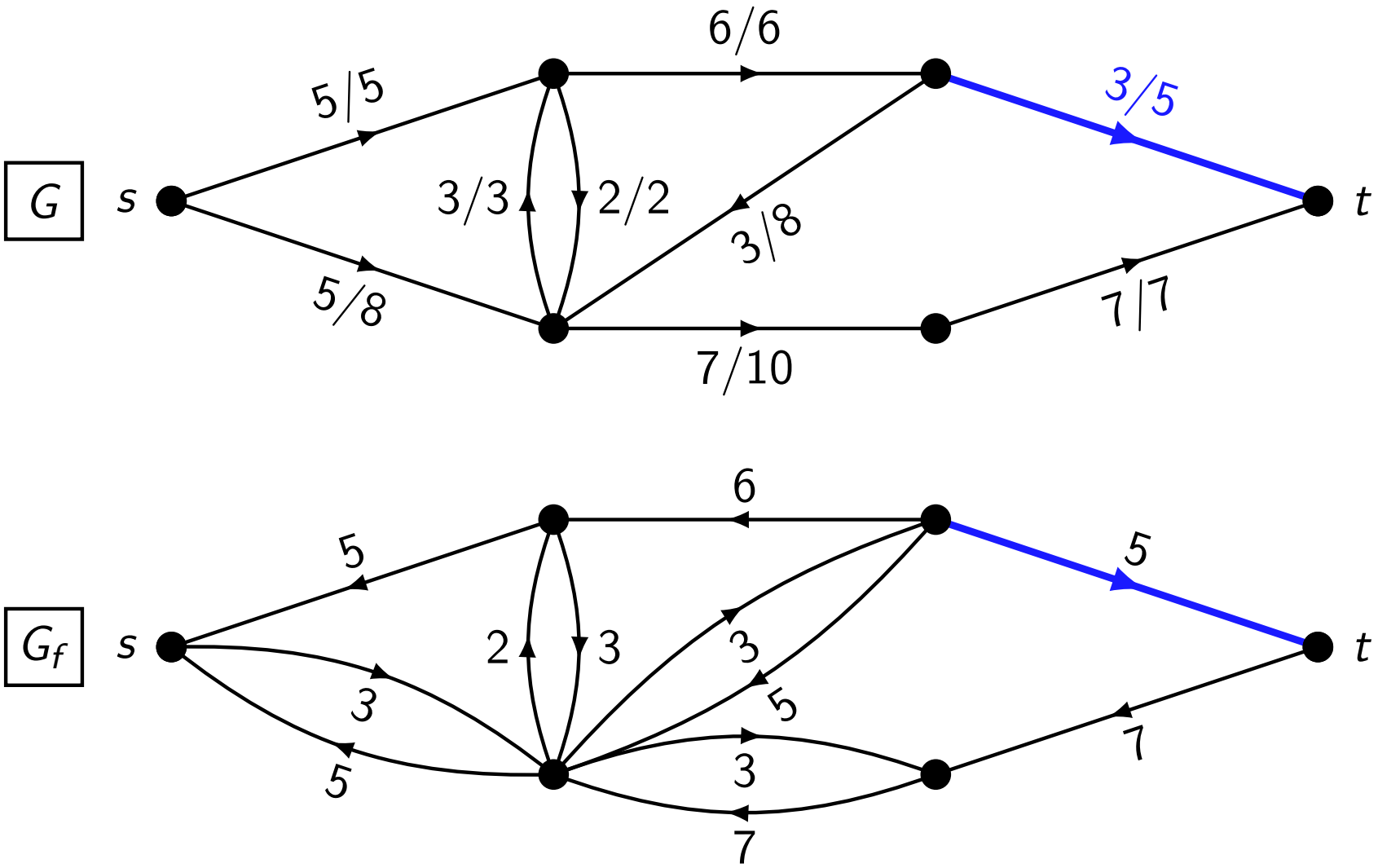

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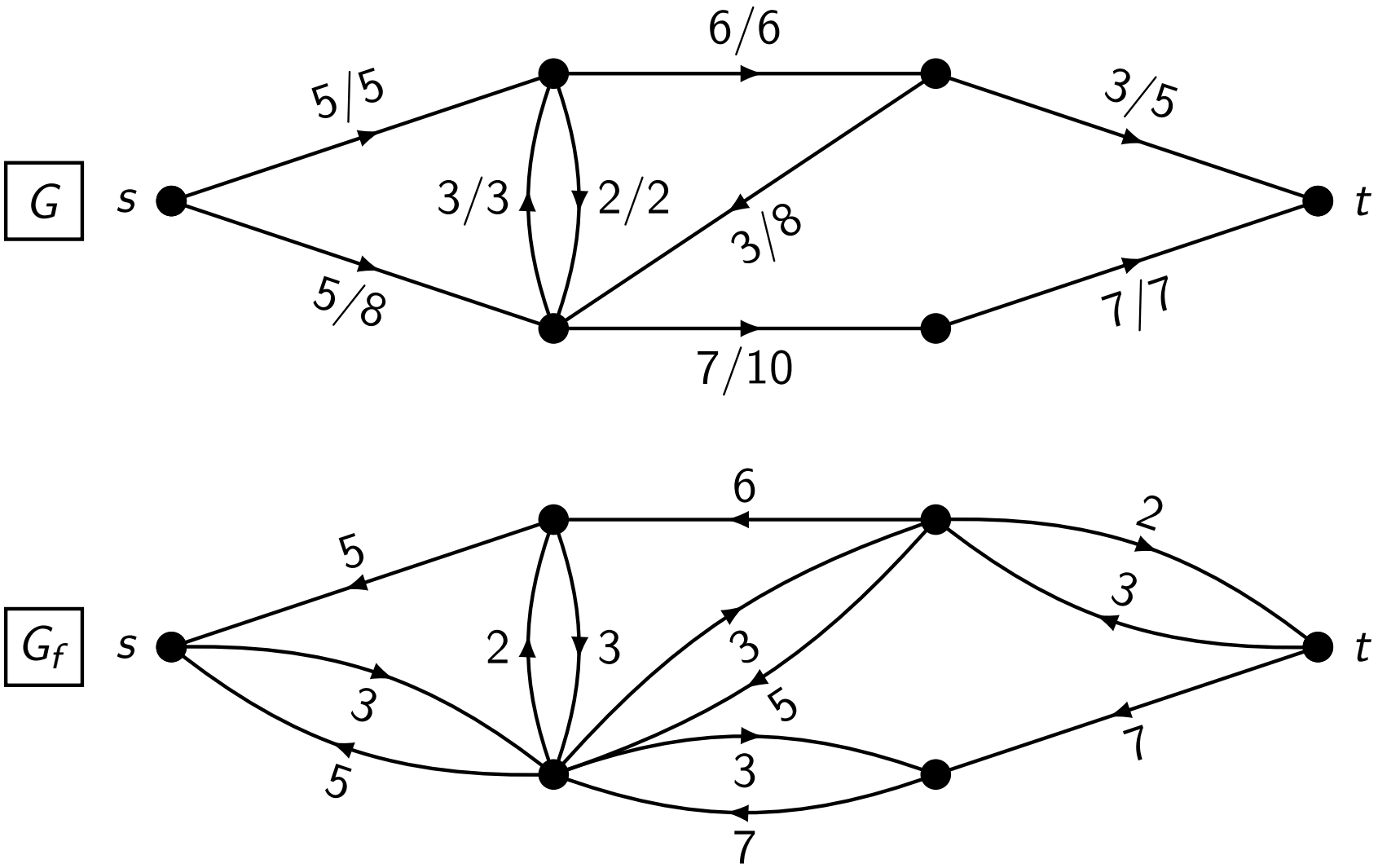

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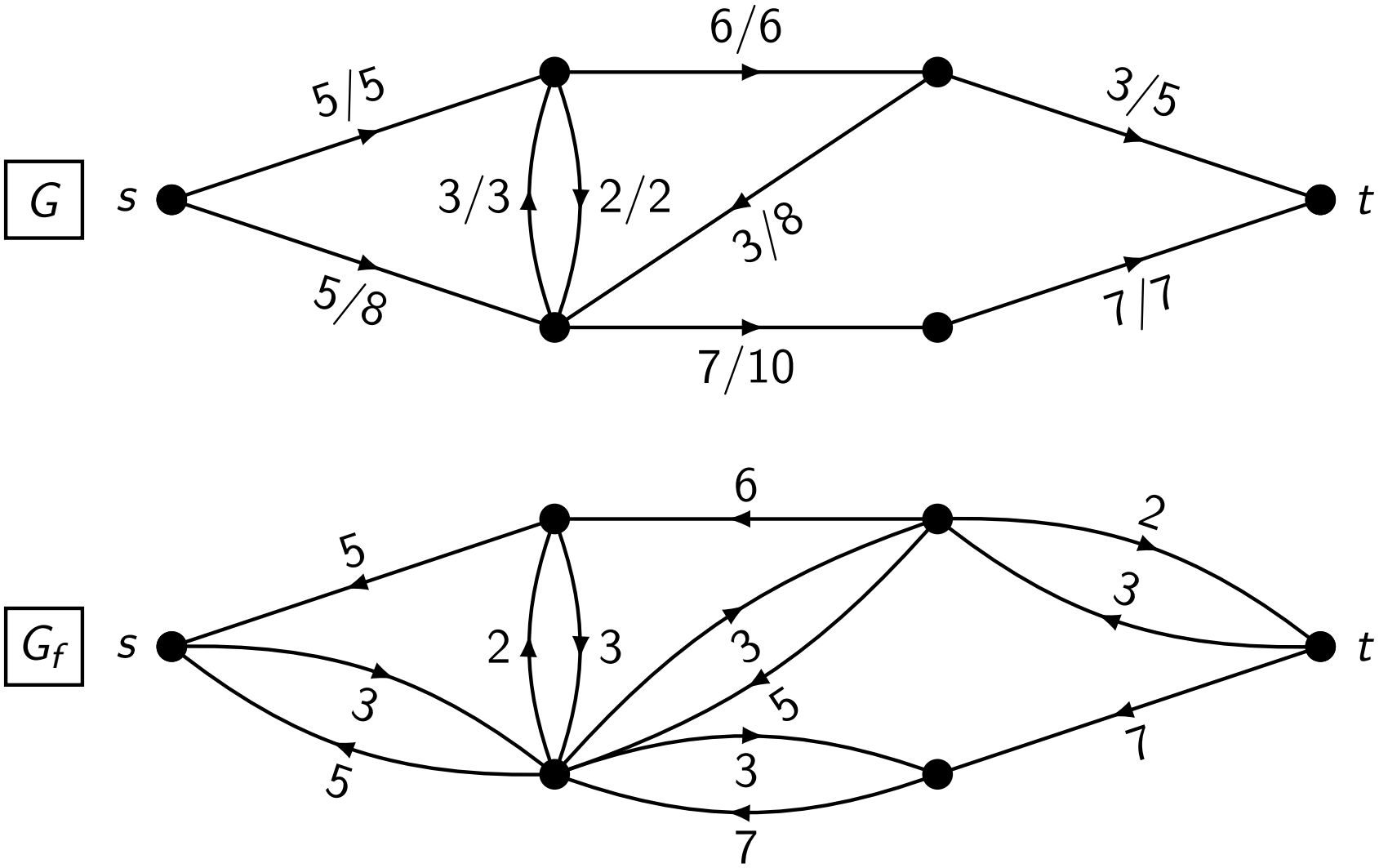
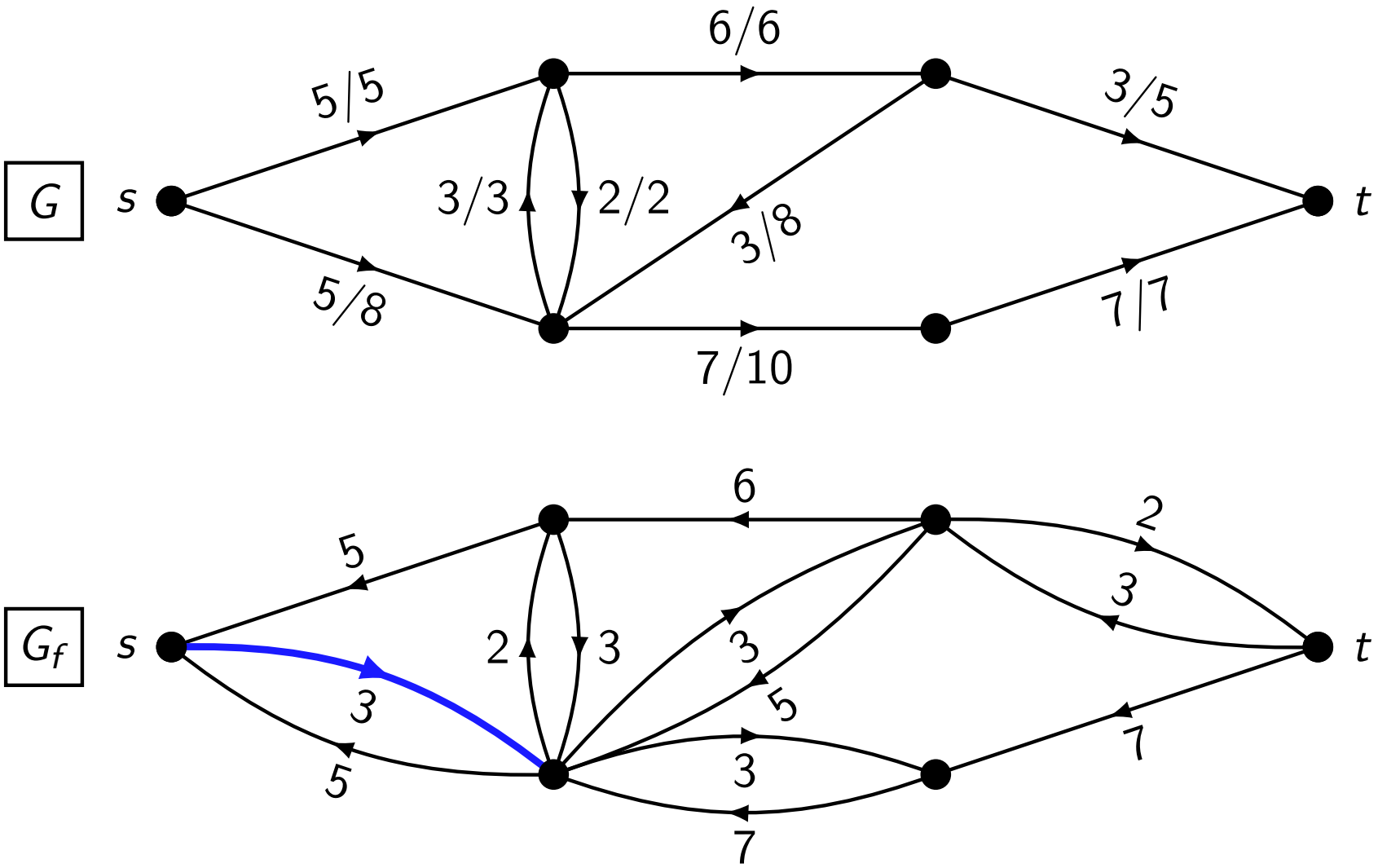


Apply depth-first search to find an augmenting path in G_f.

Worked example



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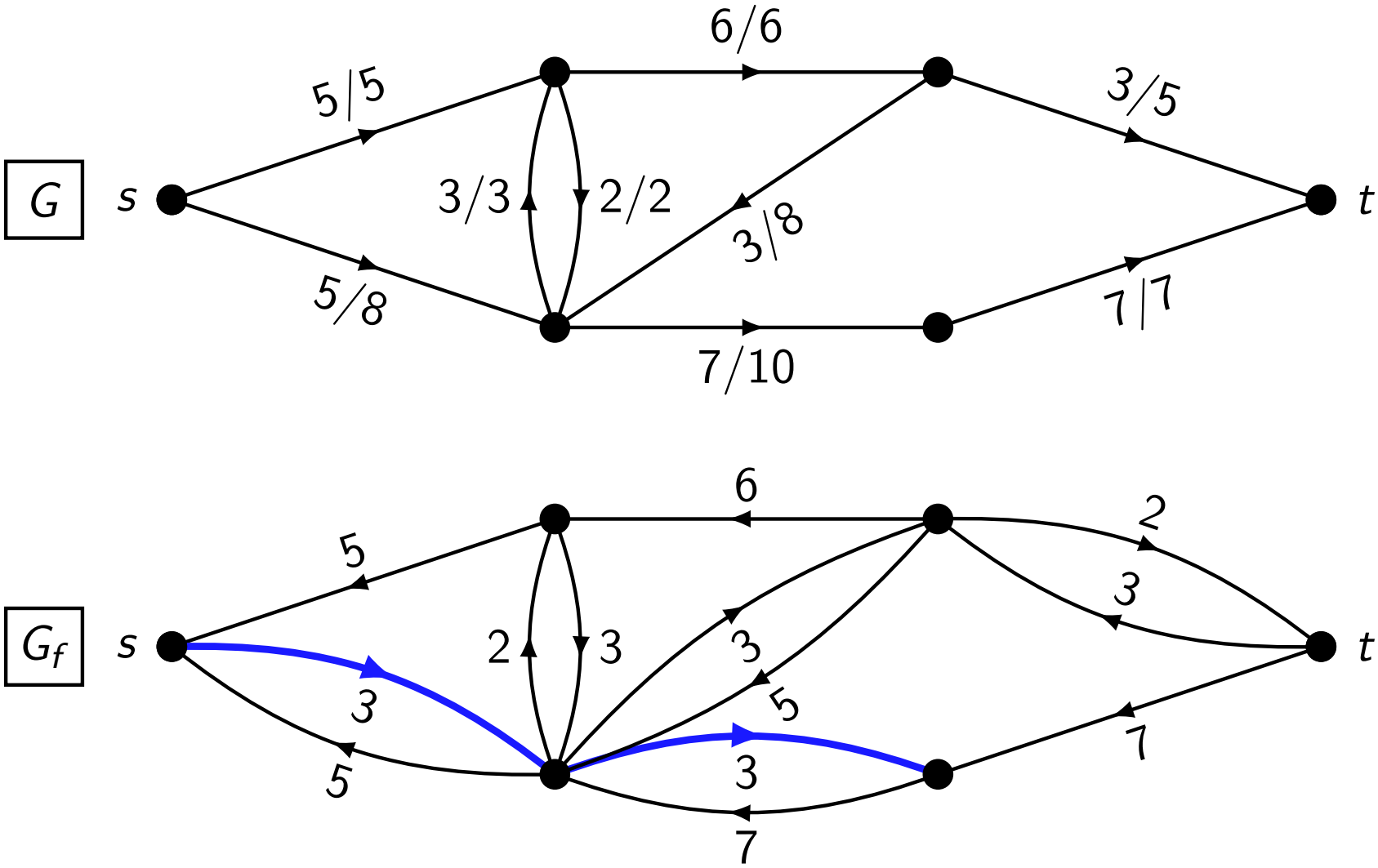

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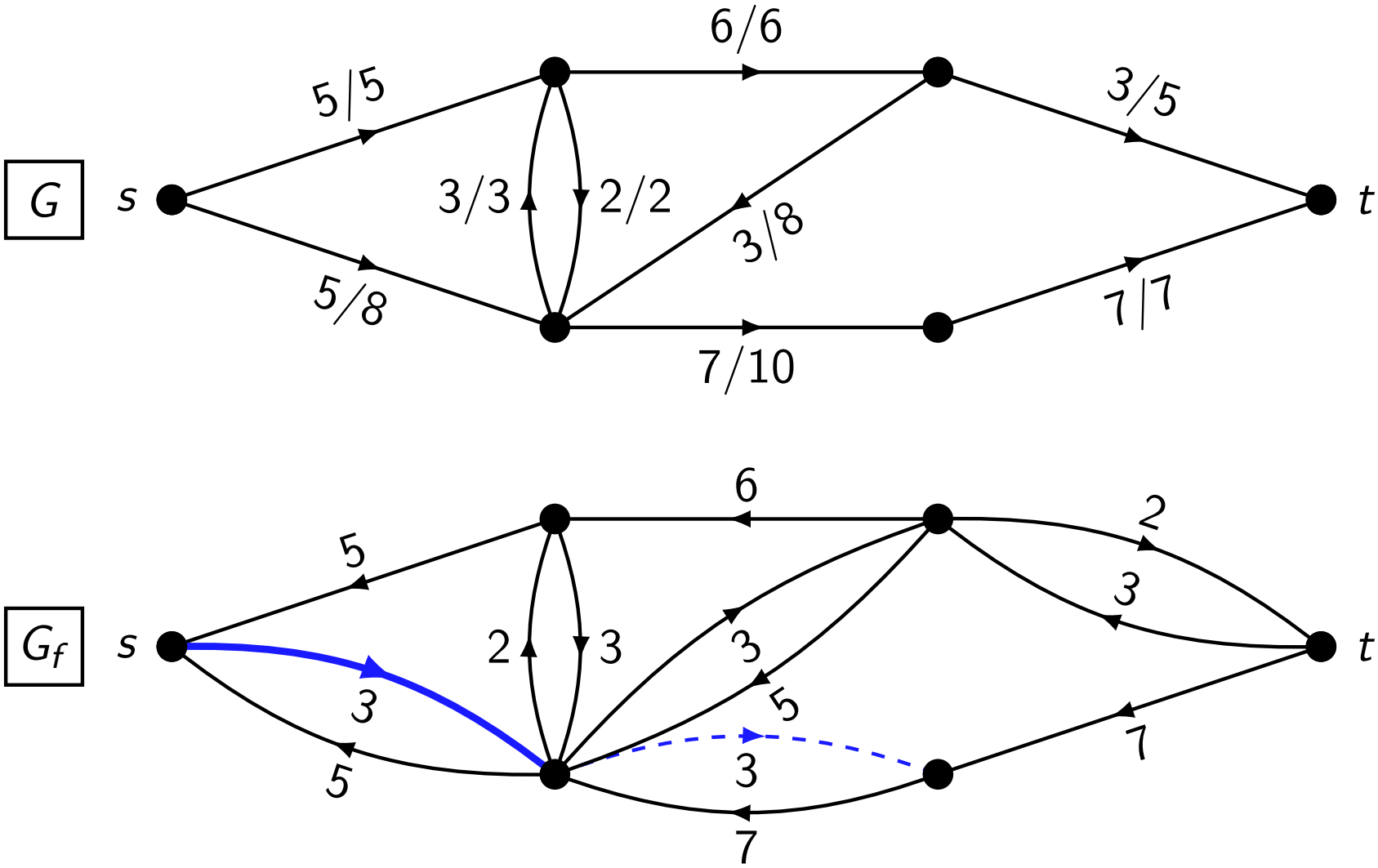

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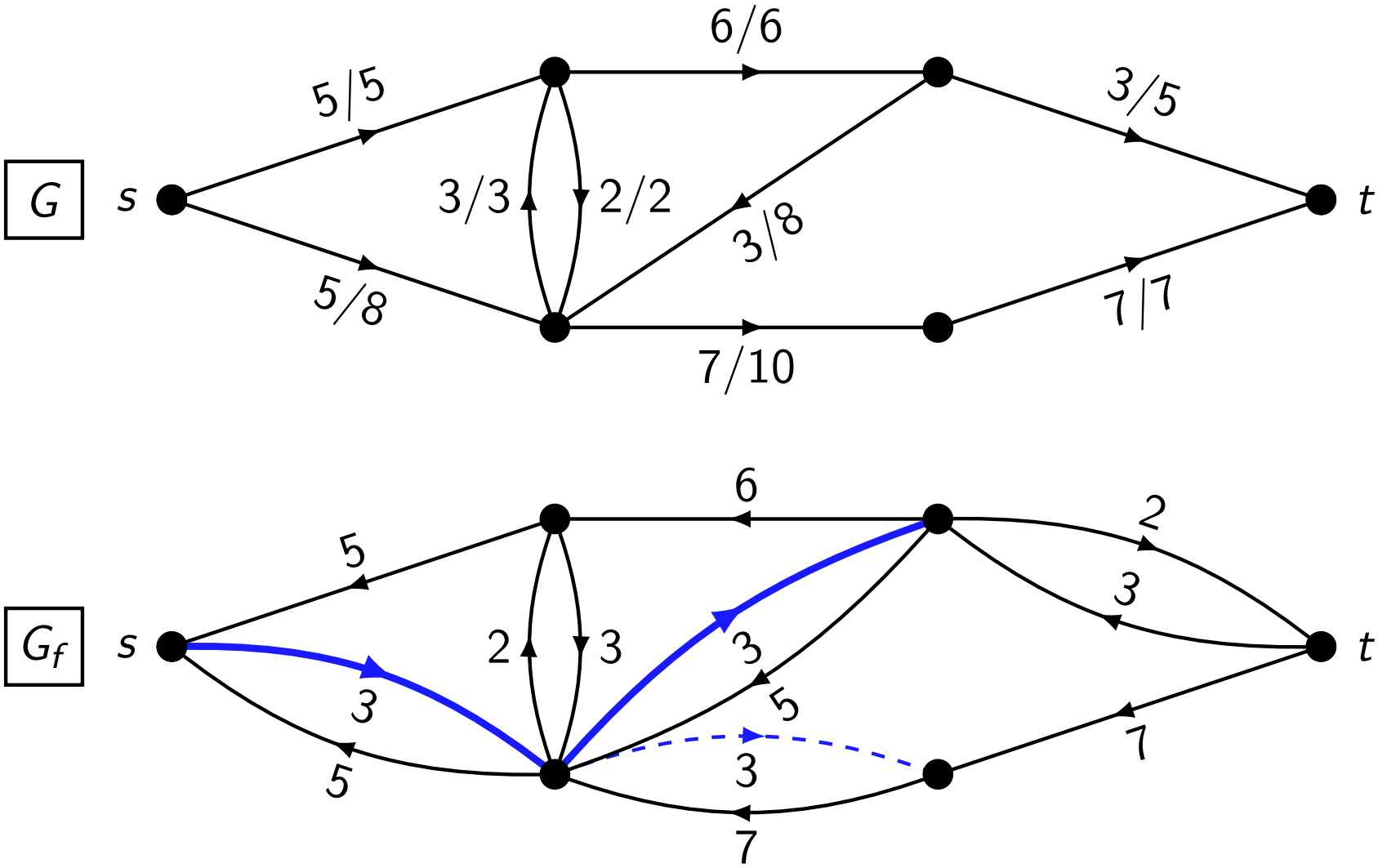

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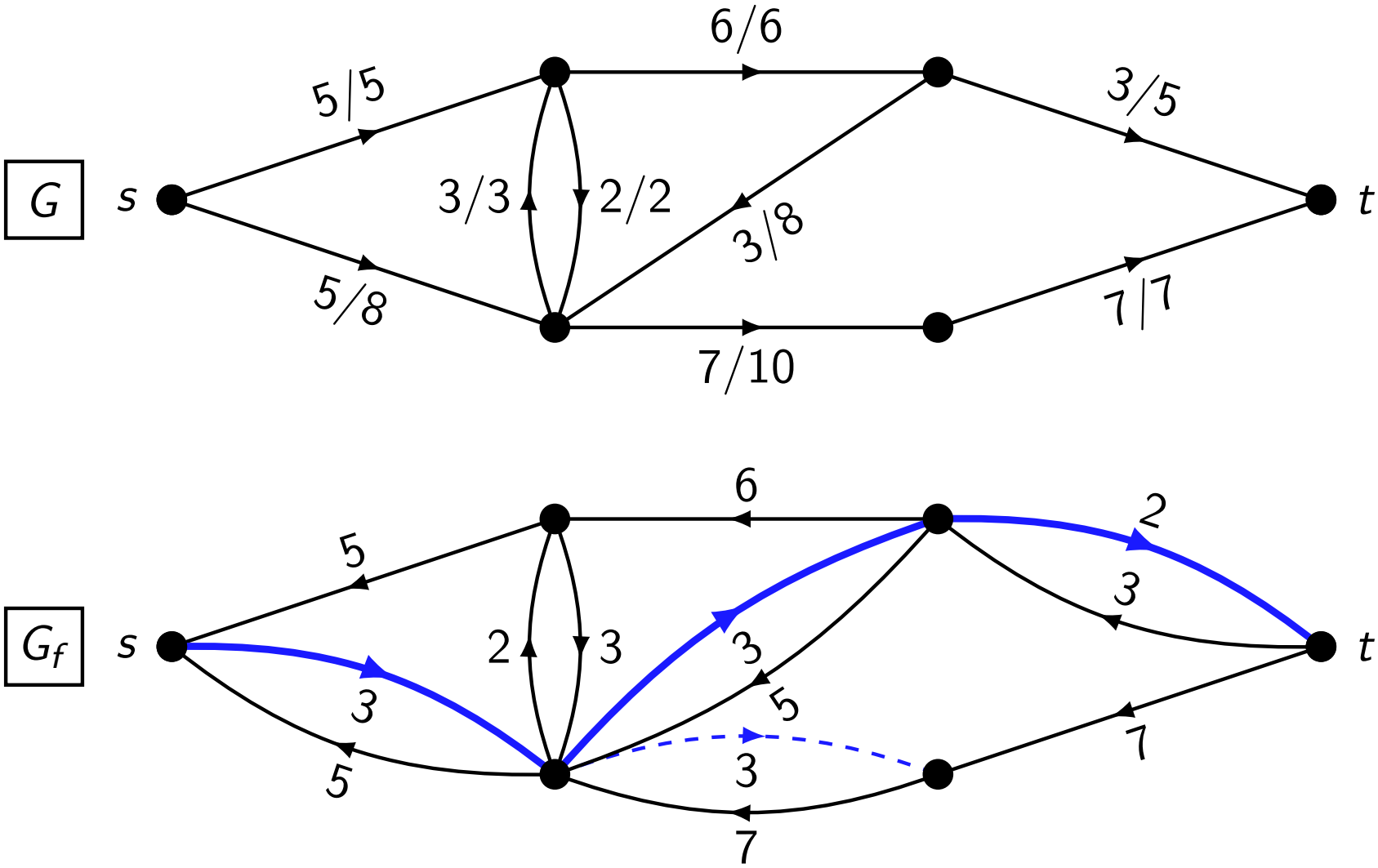

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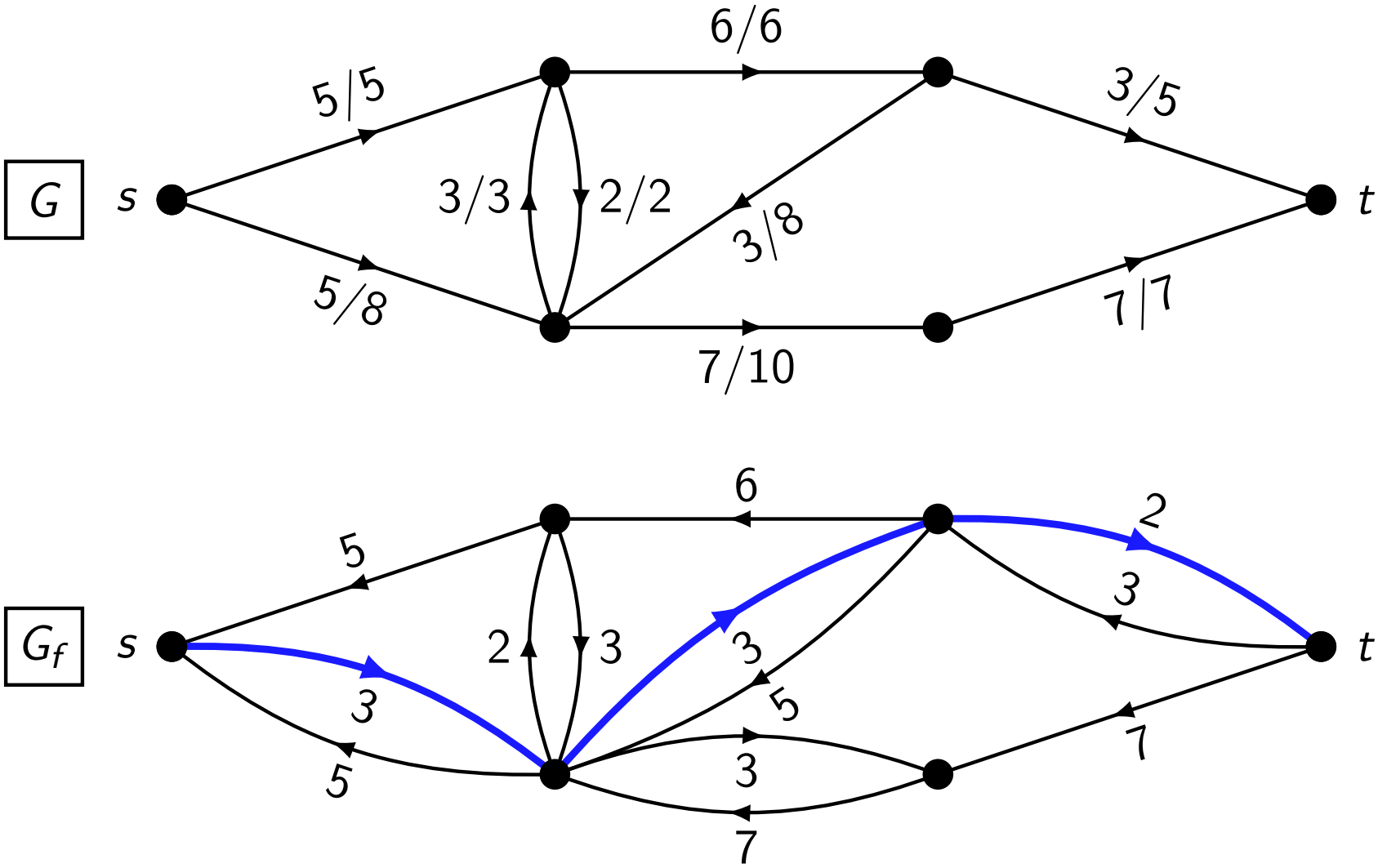

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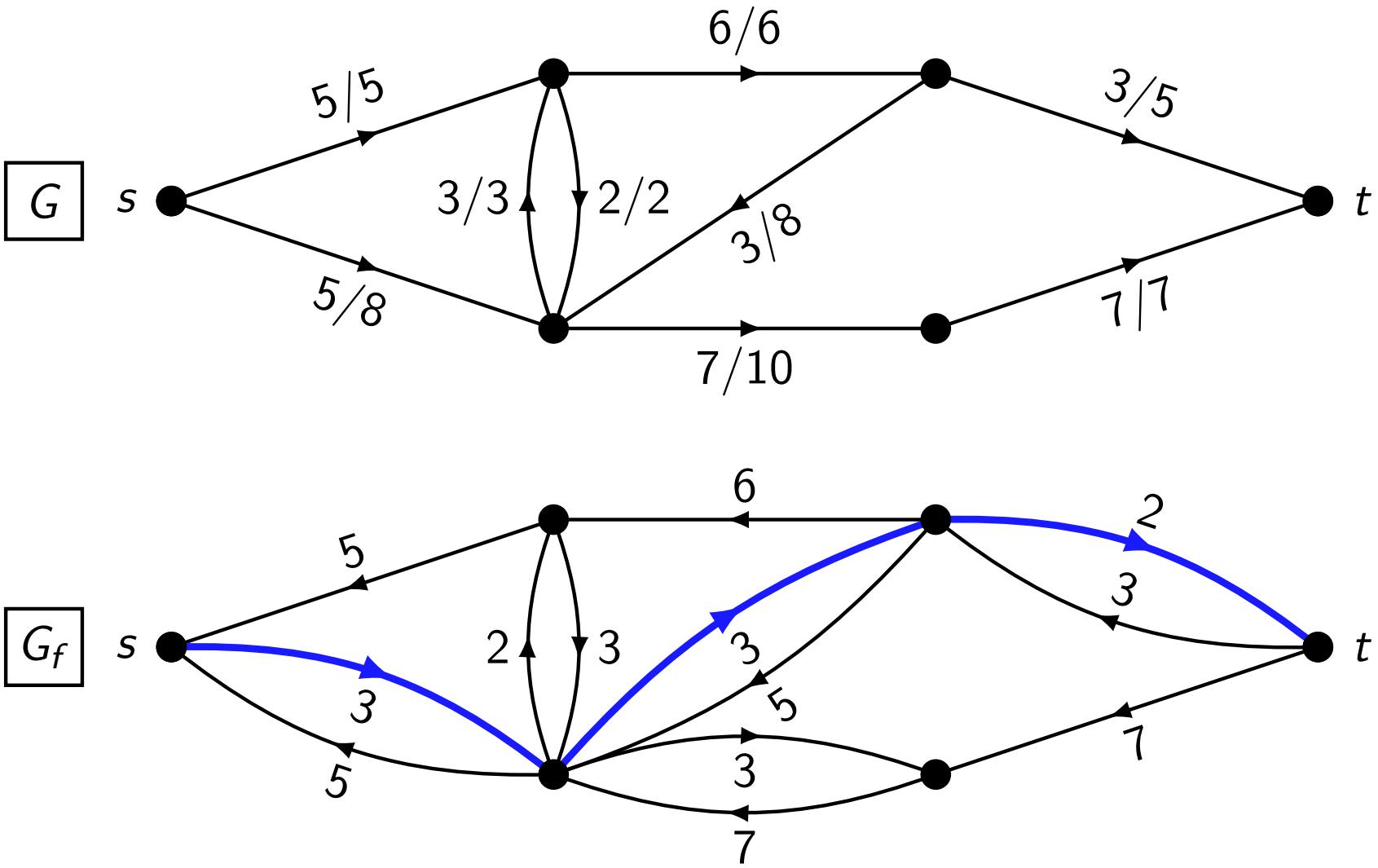
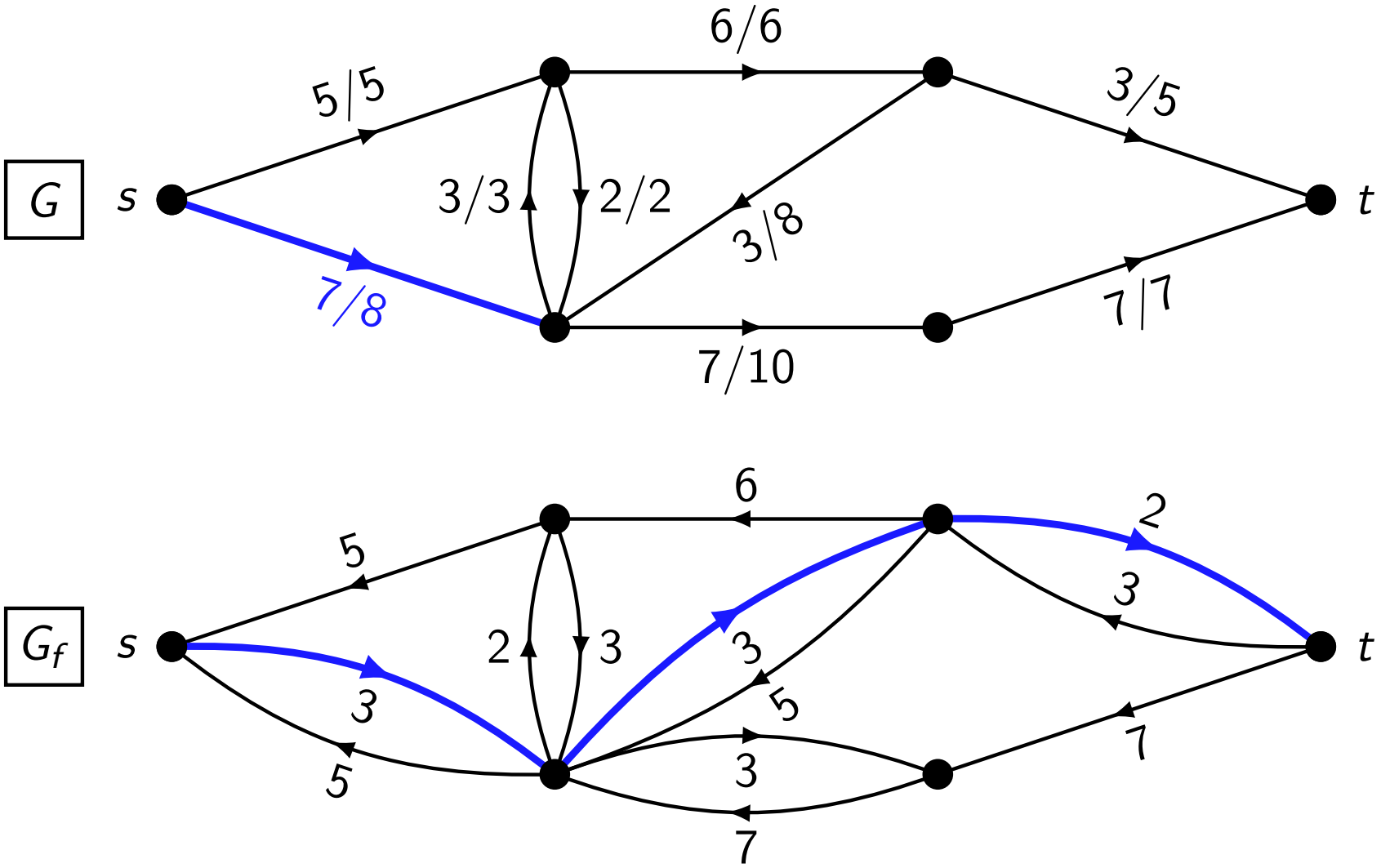


Push flow along the path. (This path has residual capacity 2.)

Worked example



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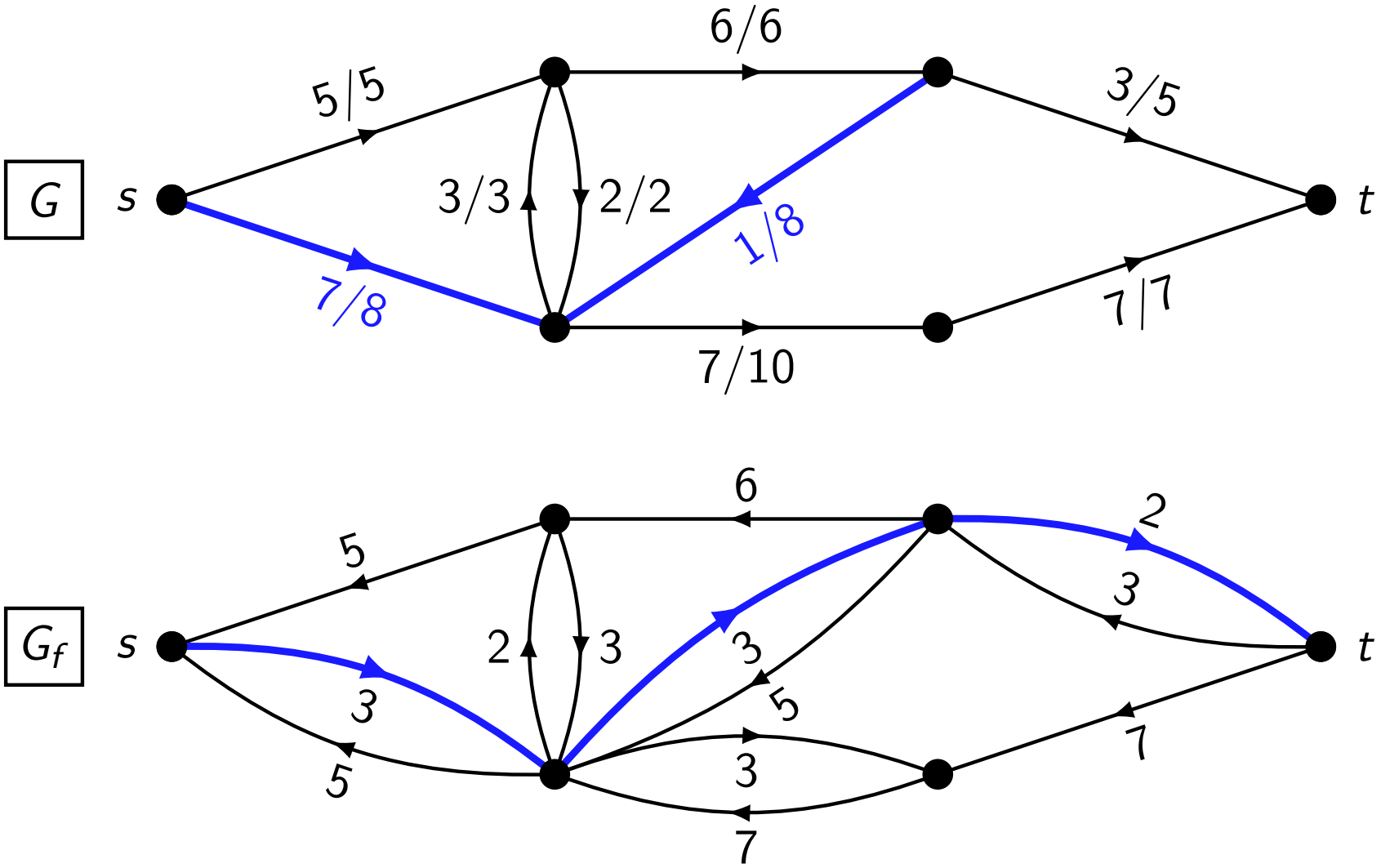

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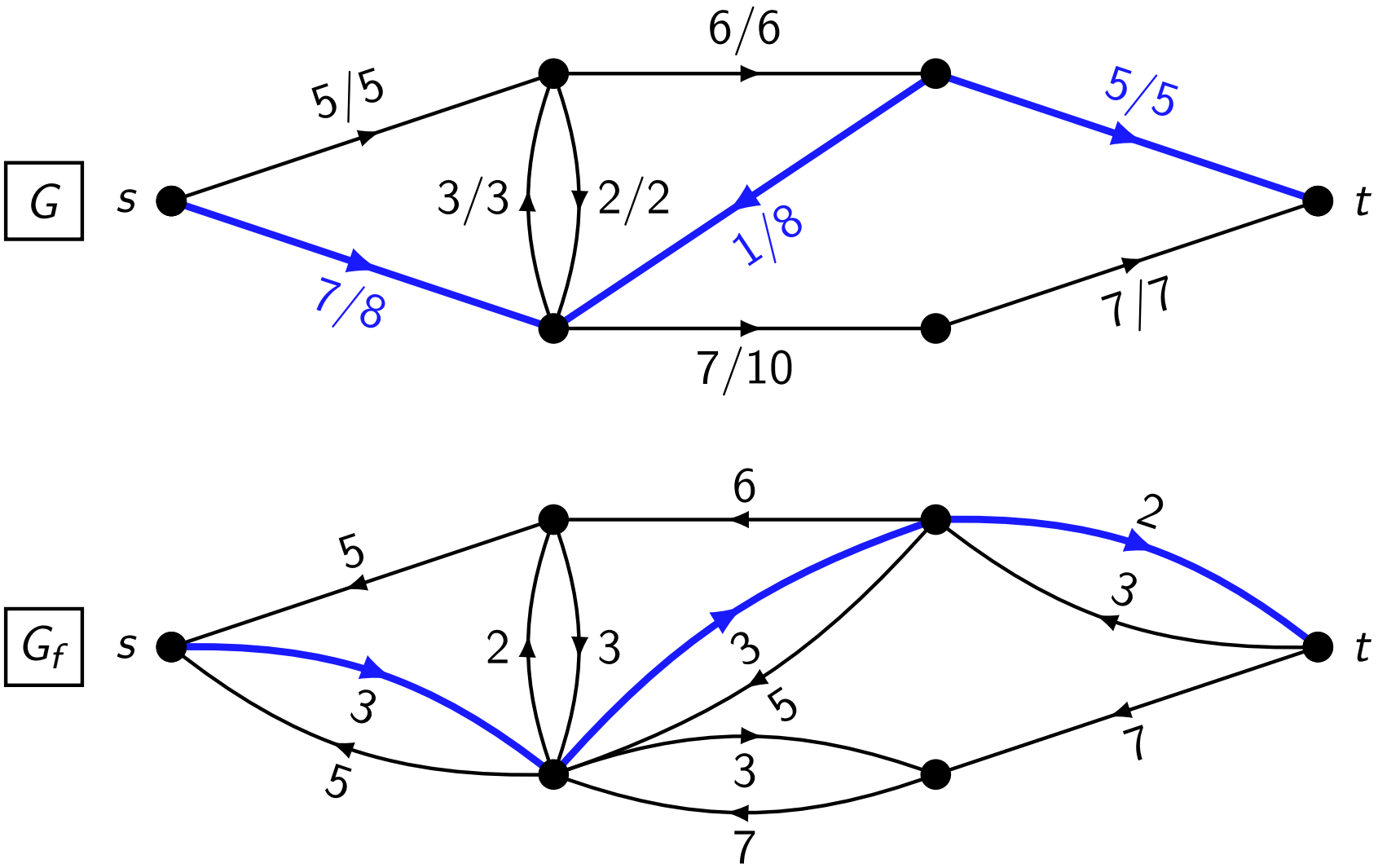

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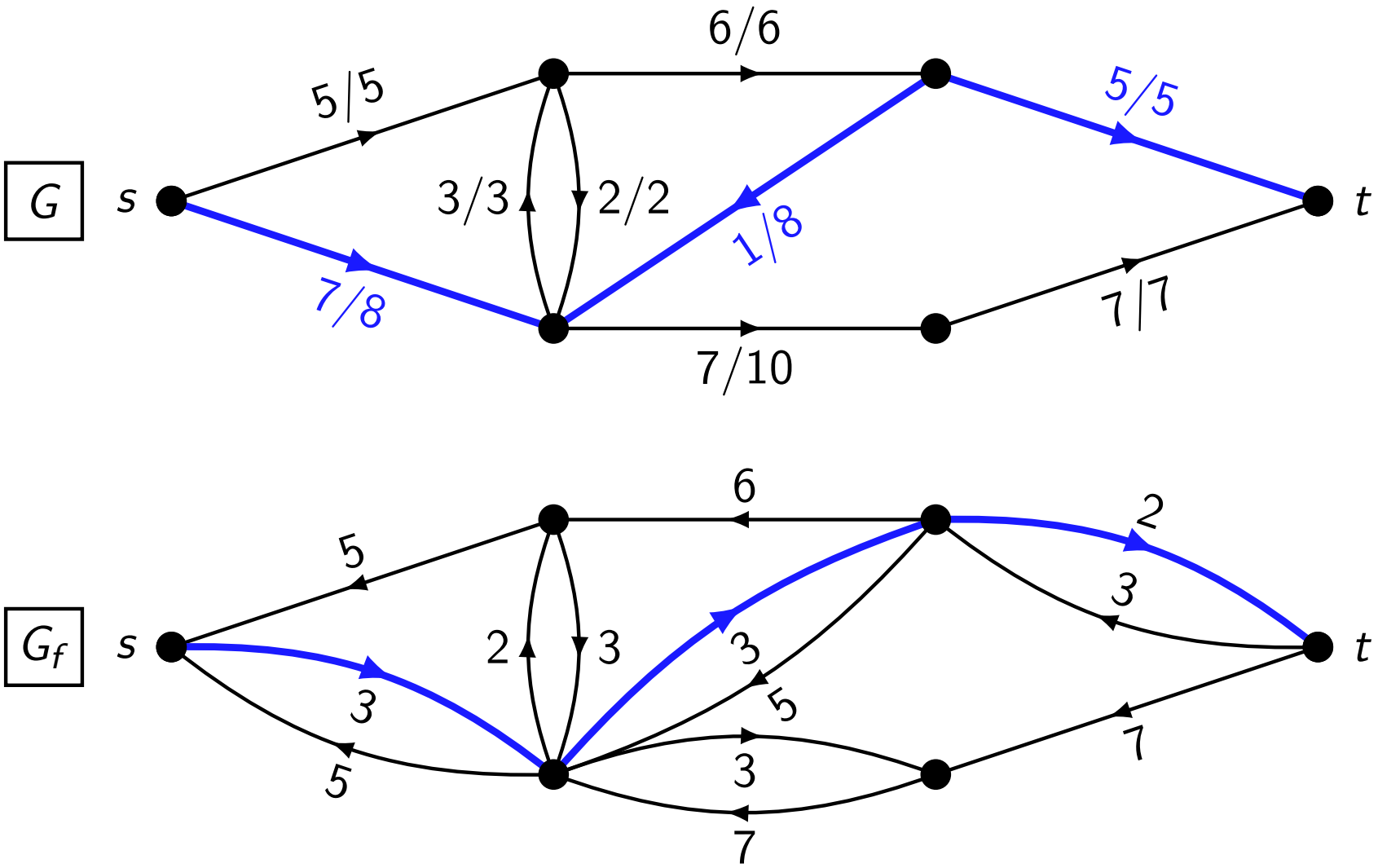
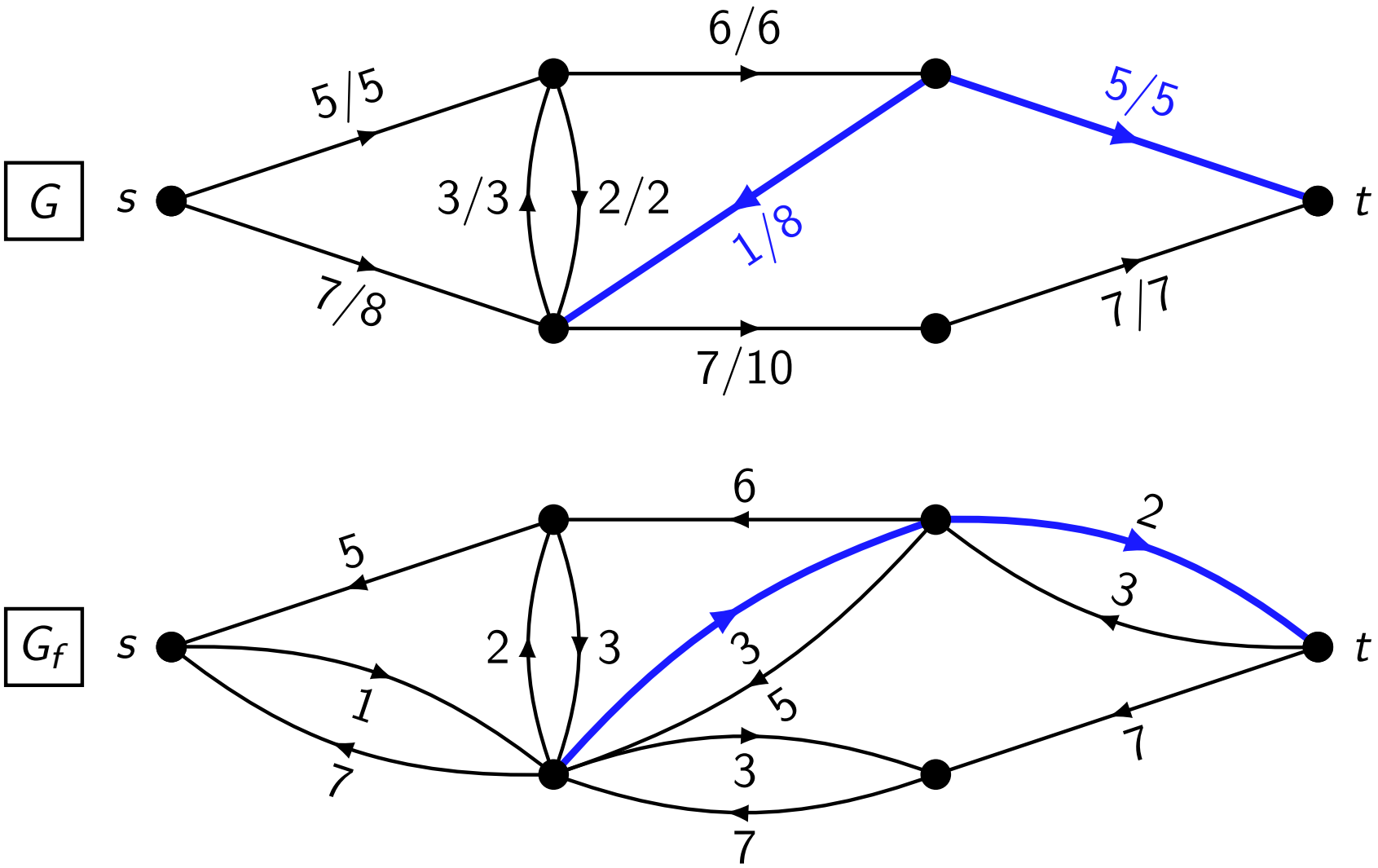


Update G_f along the path.

Worked example



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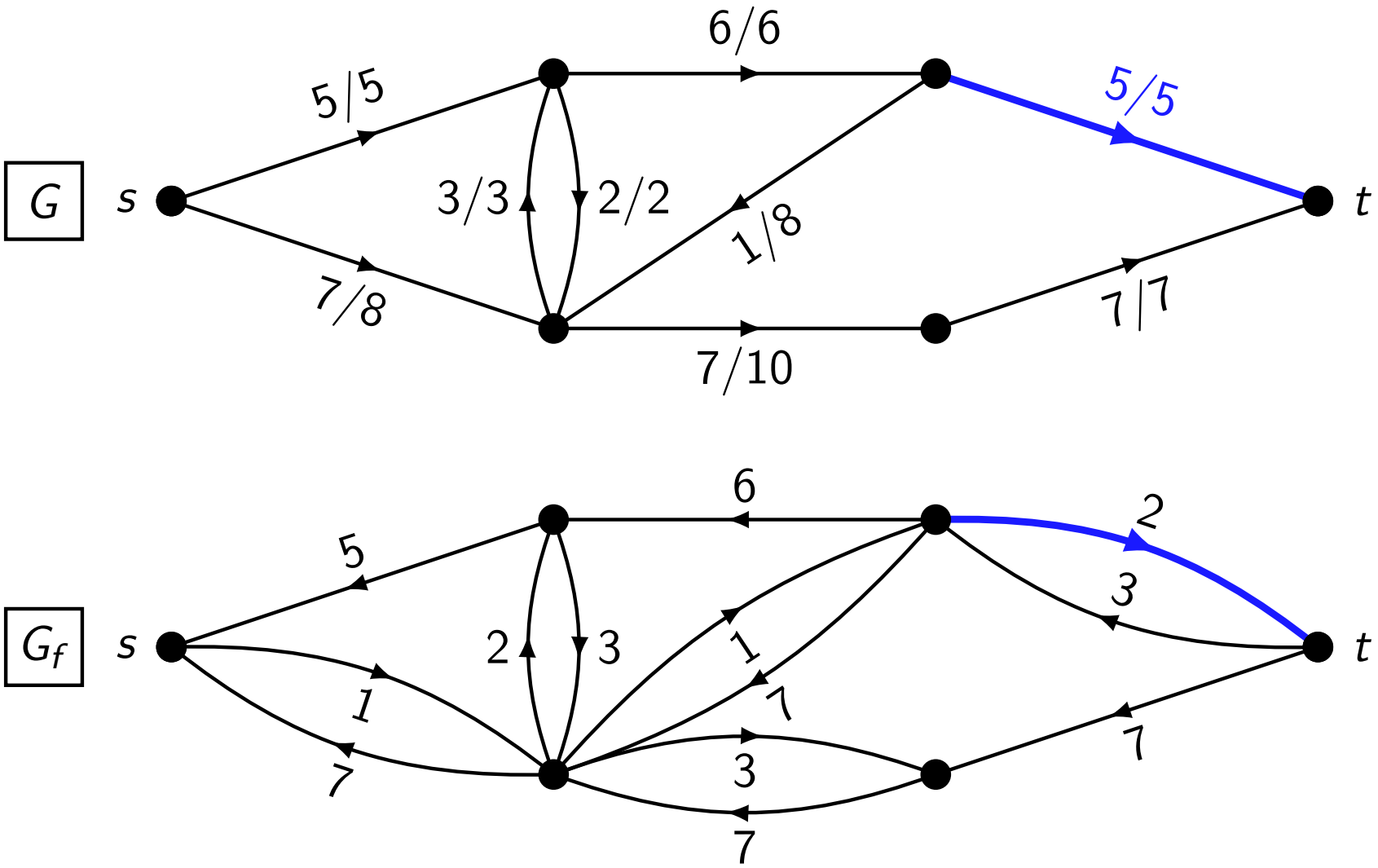

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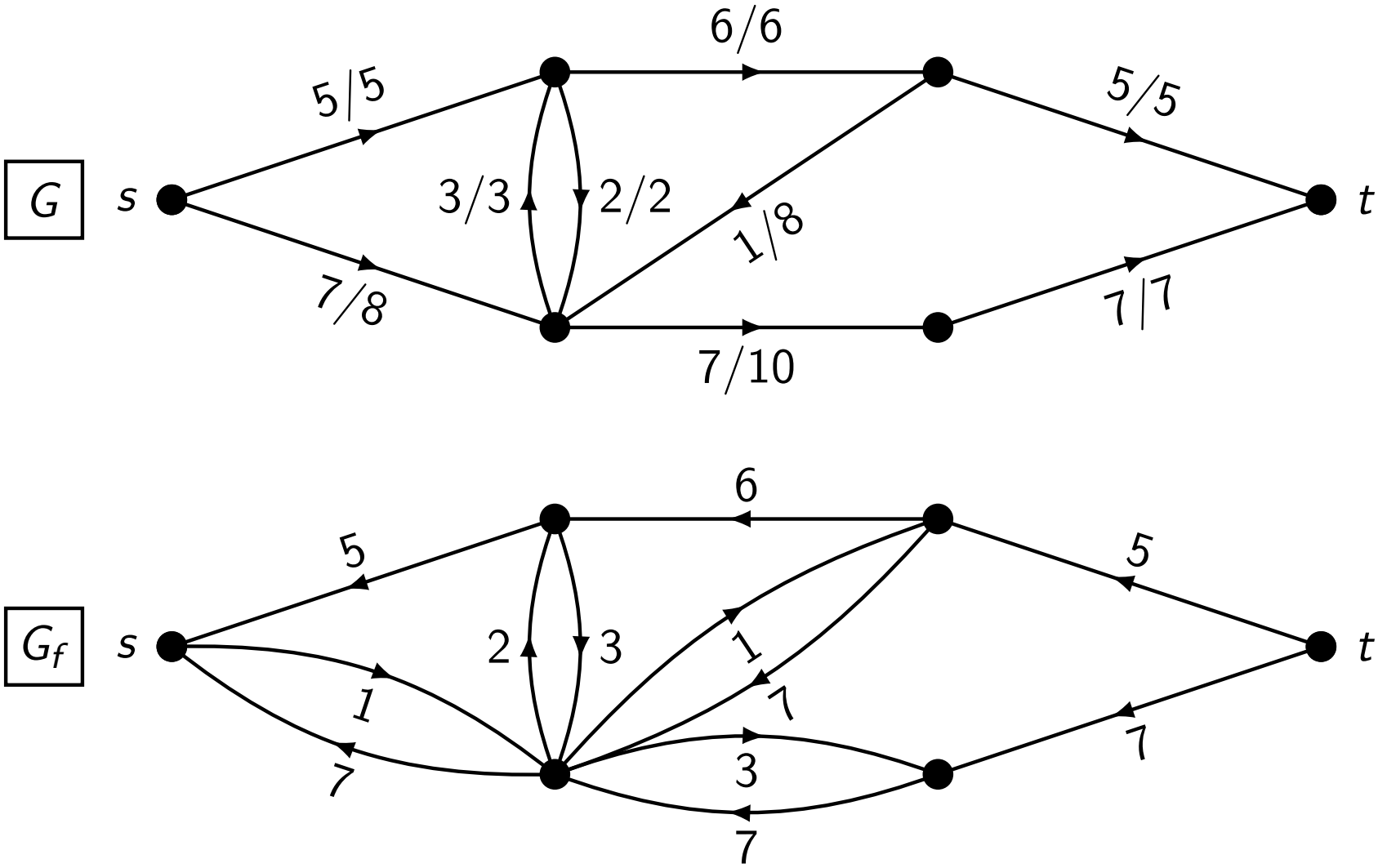

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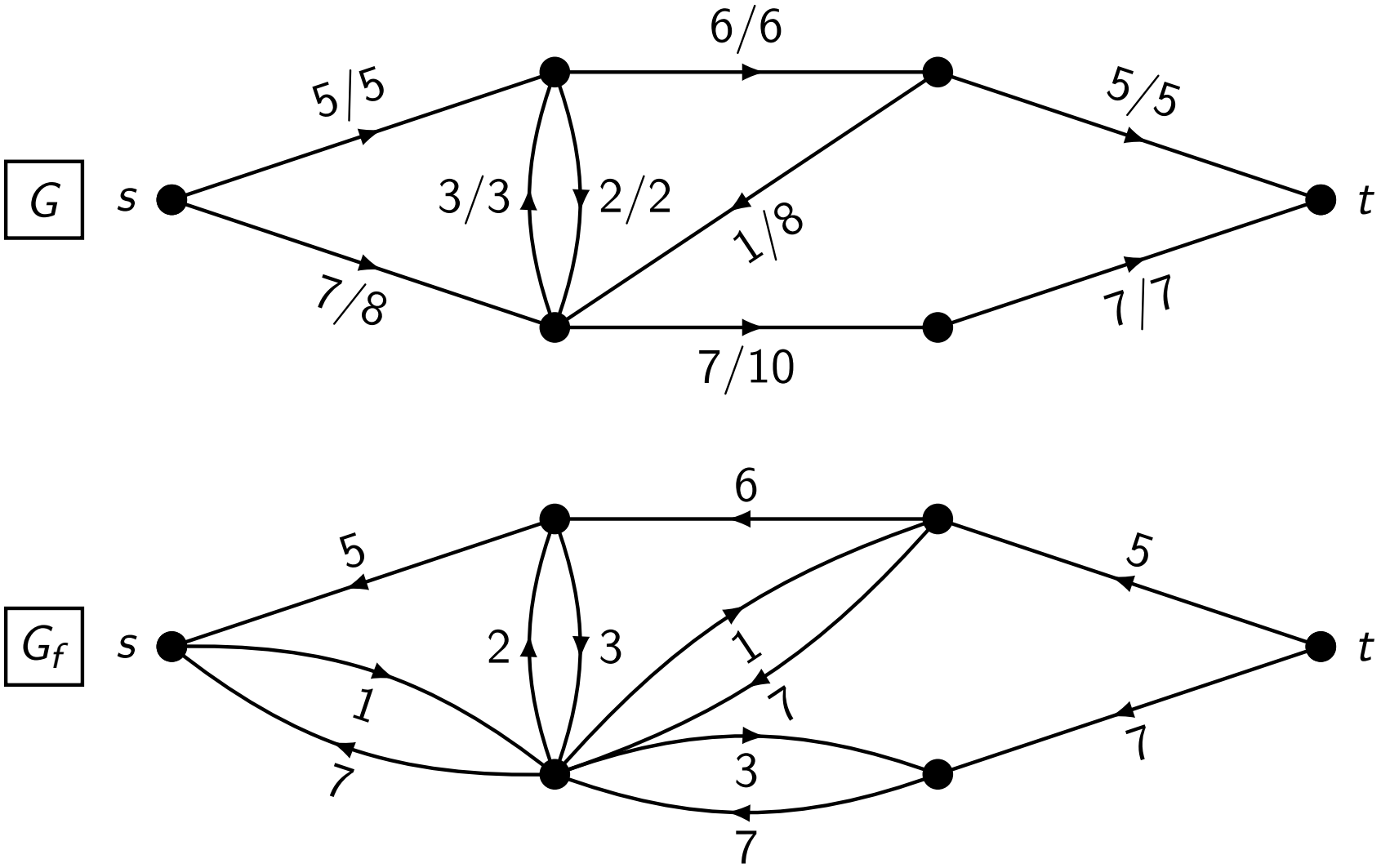

Apply depth-first search to find an augmenting path in G_f.

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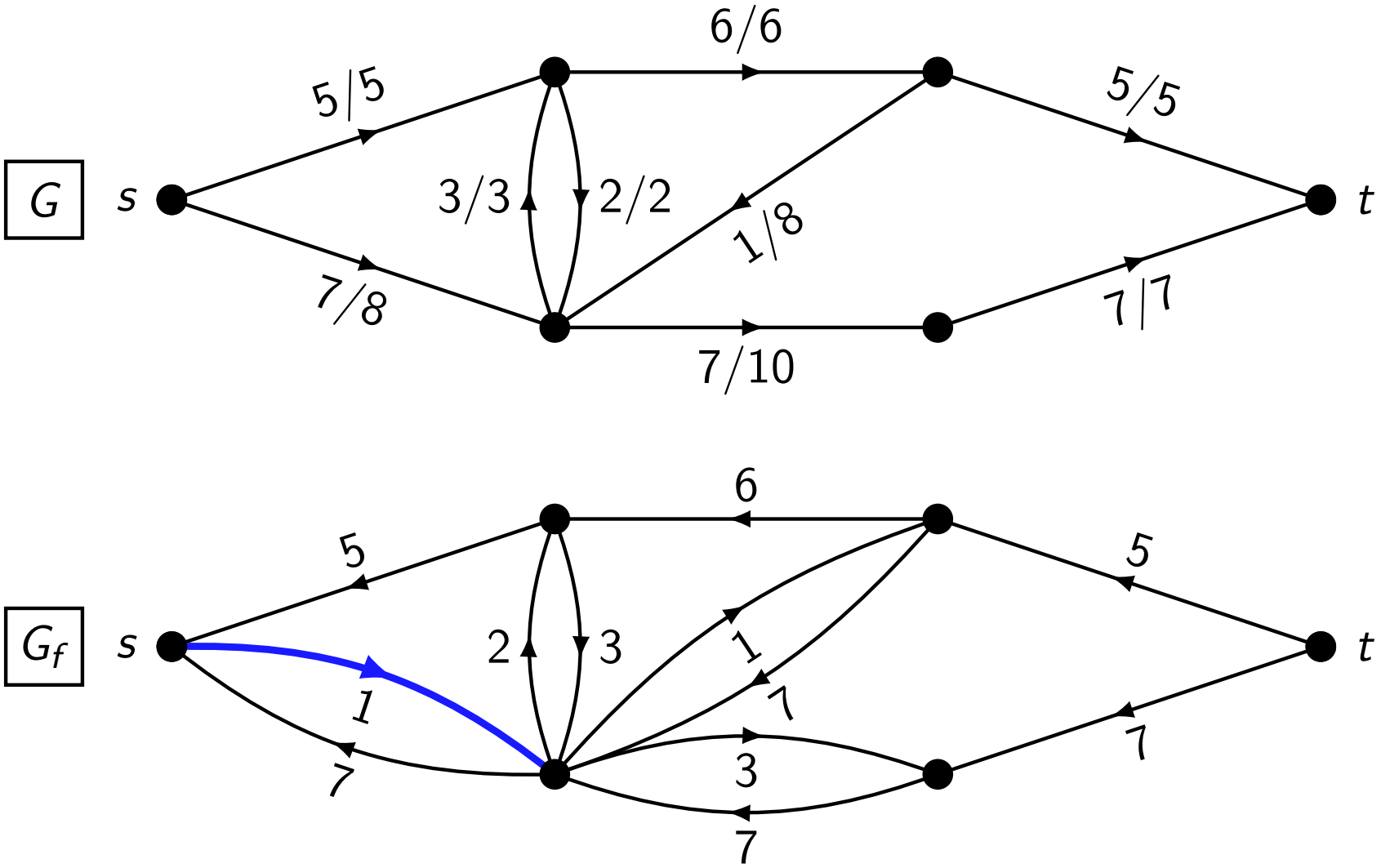

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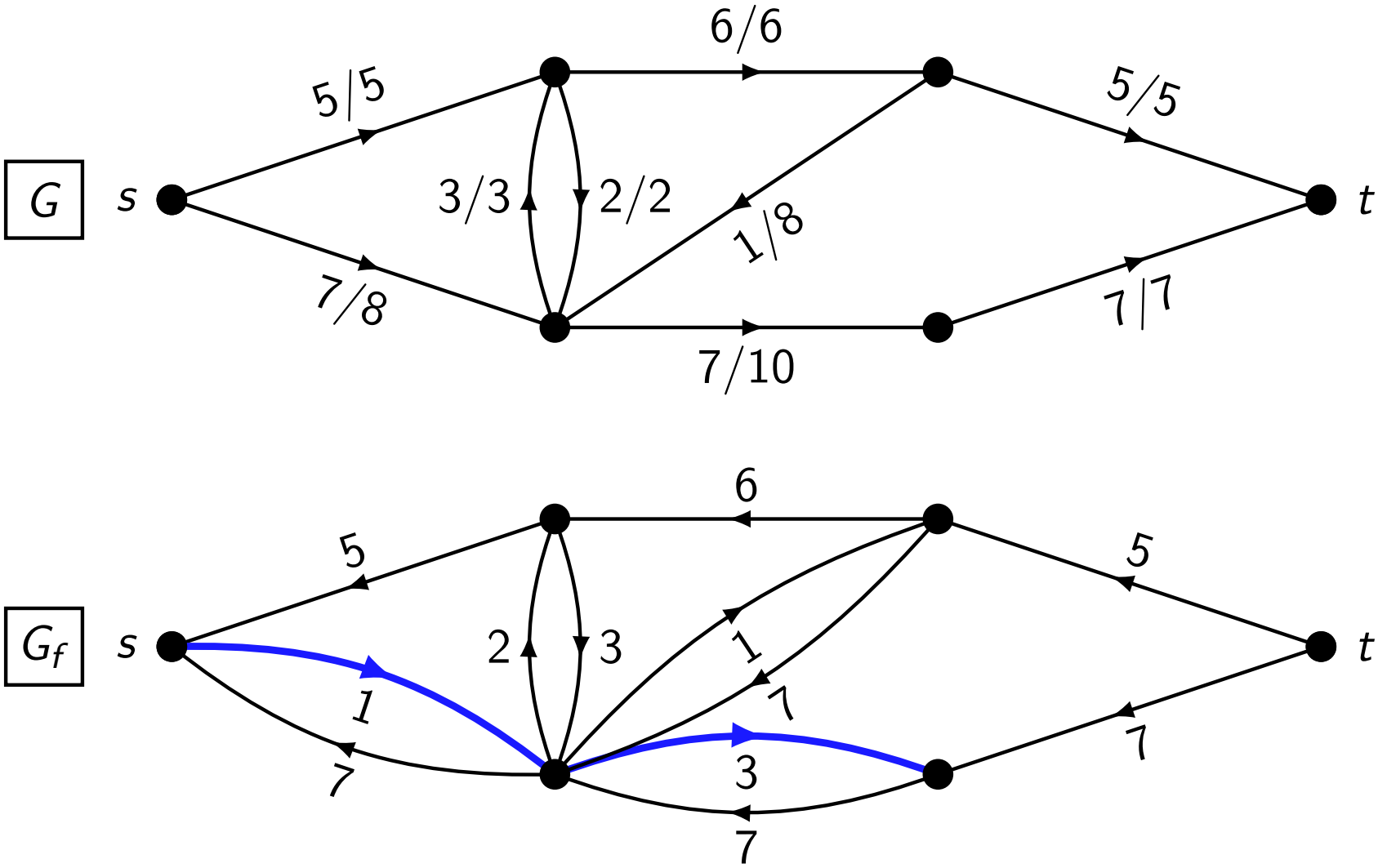

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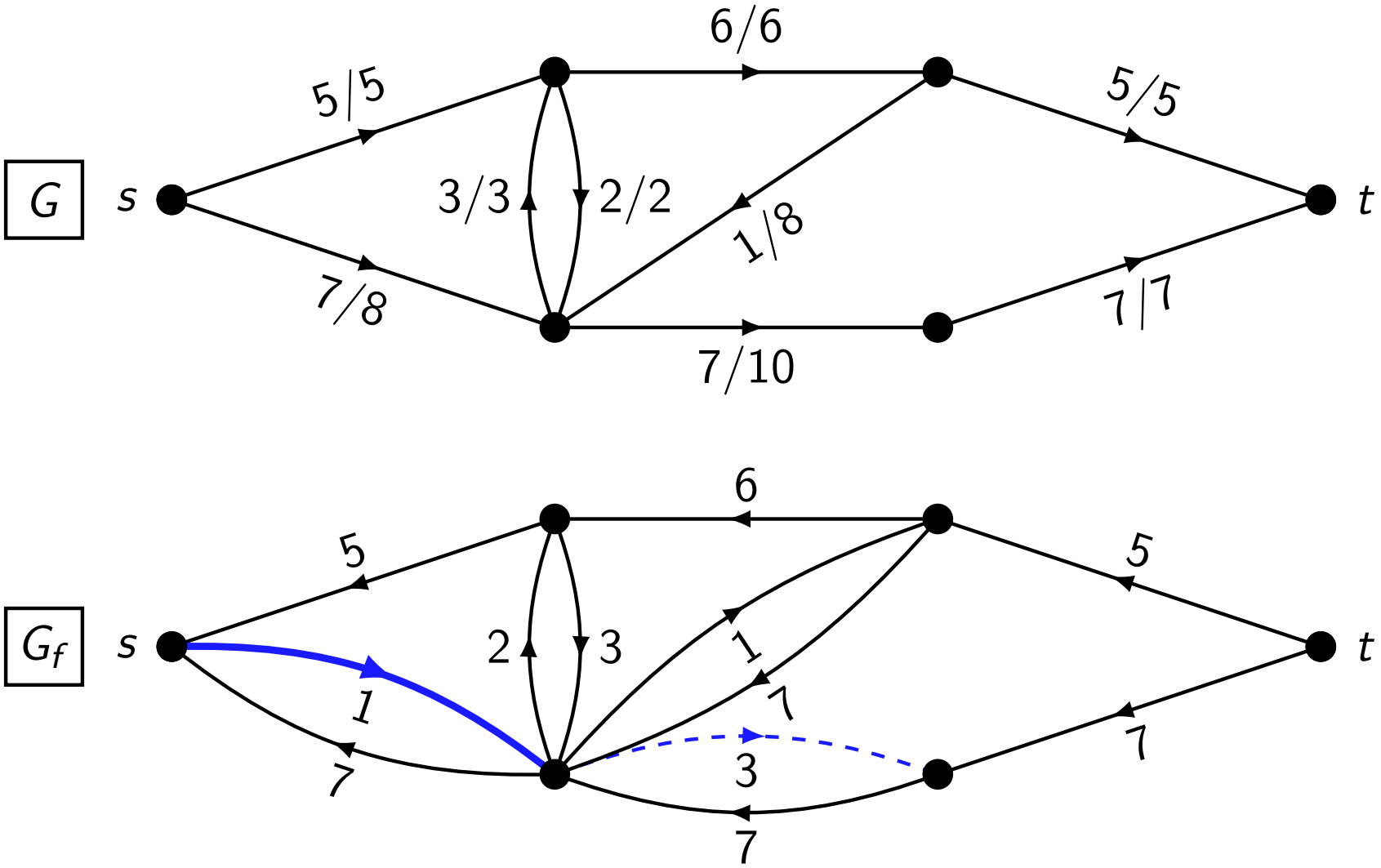

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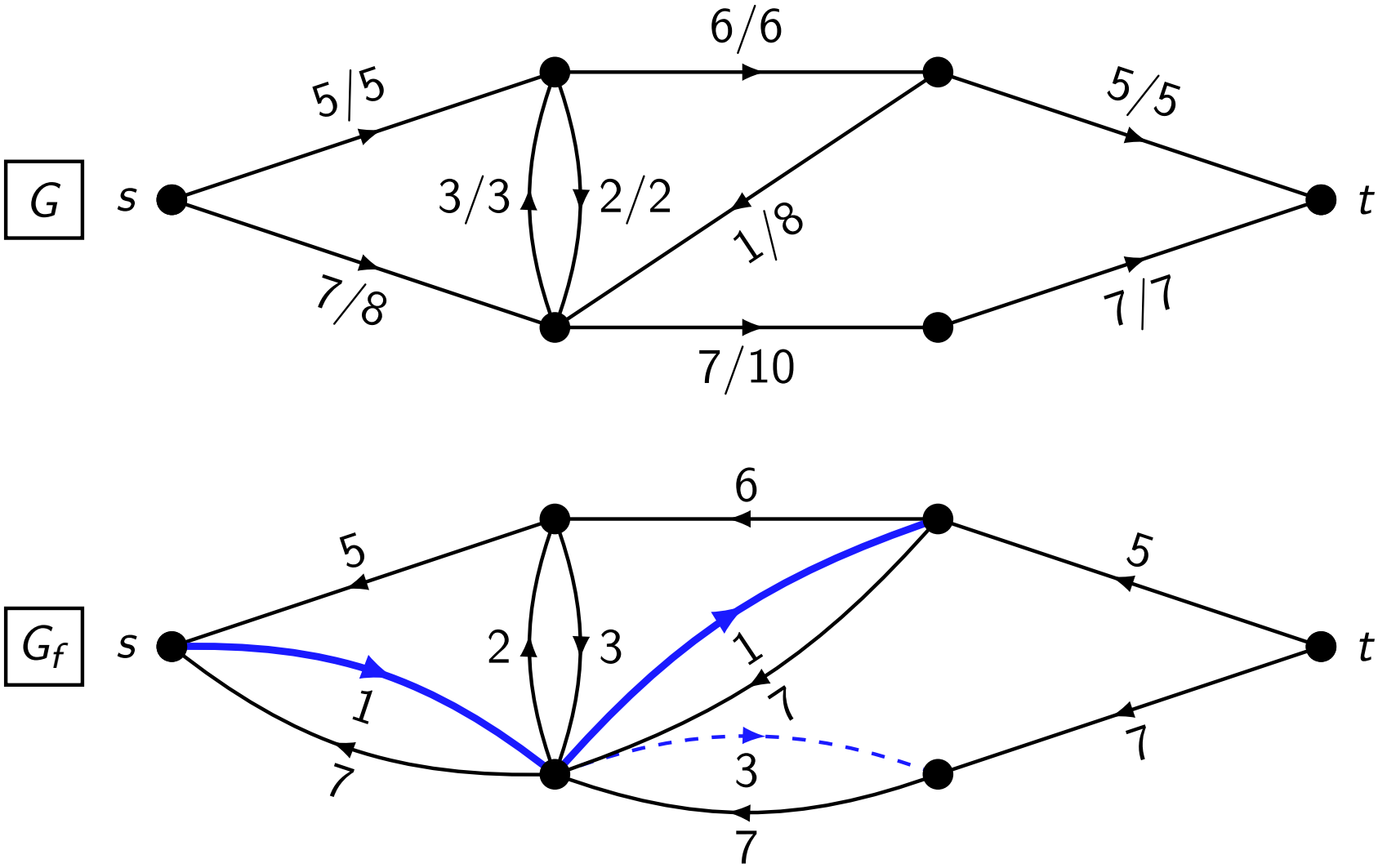

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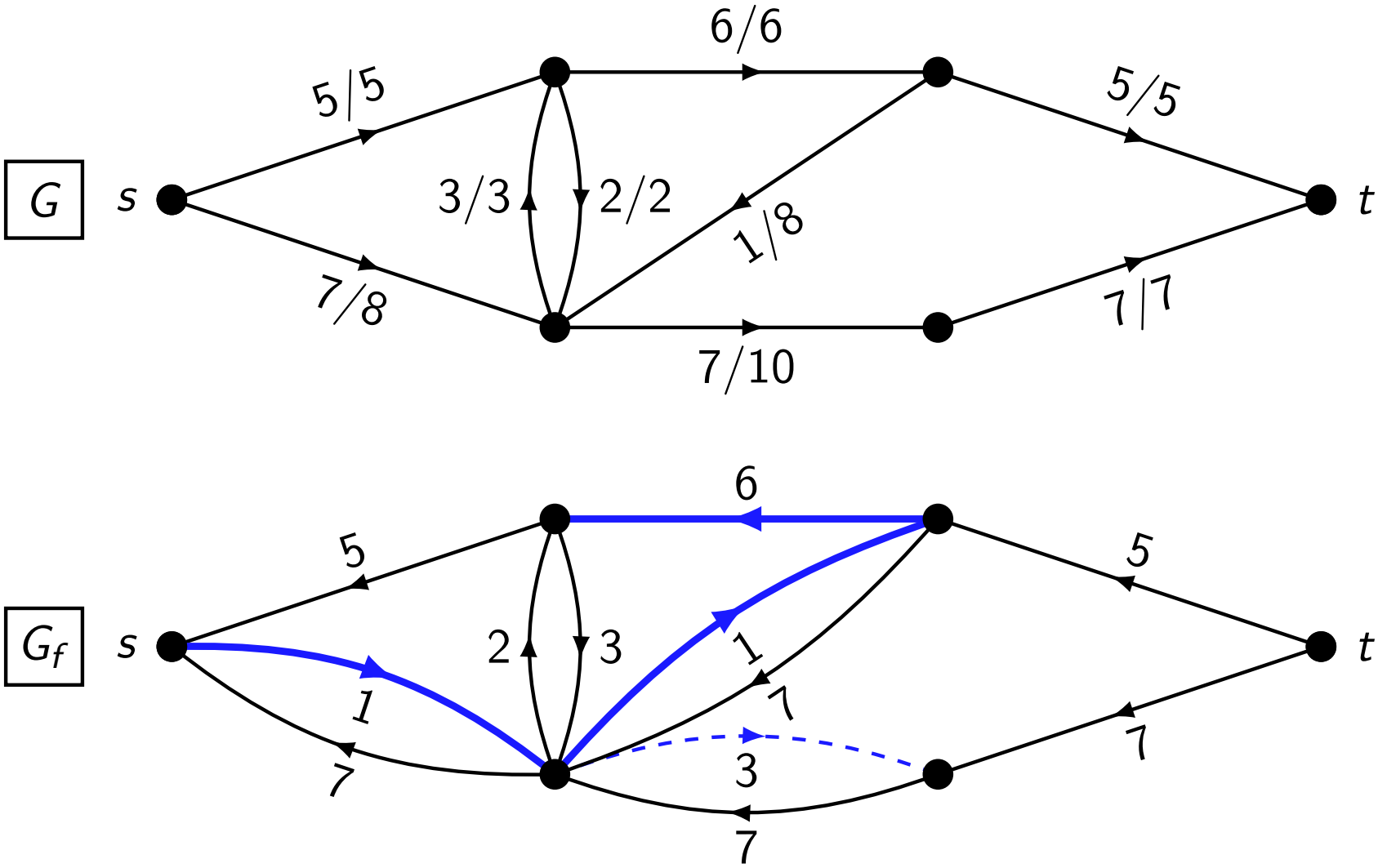

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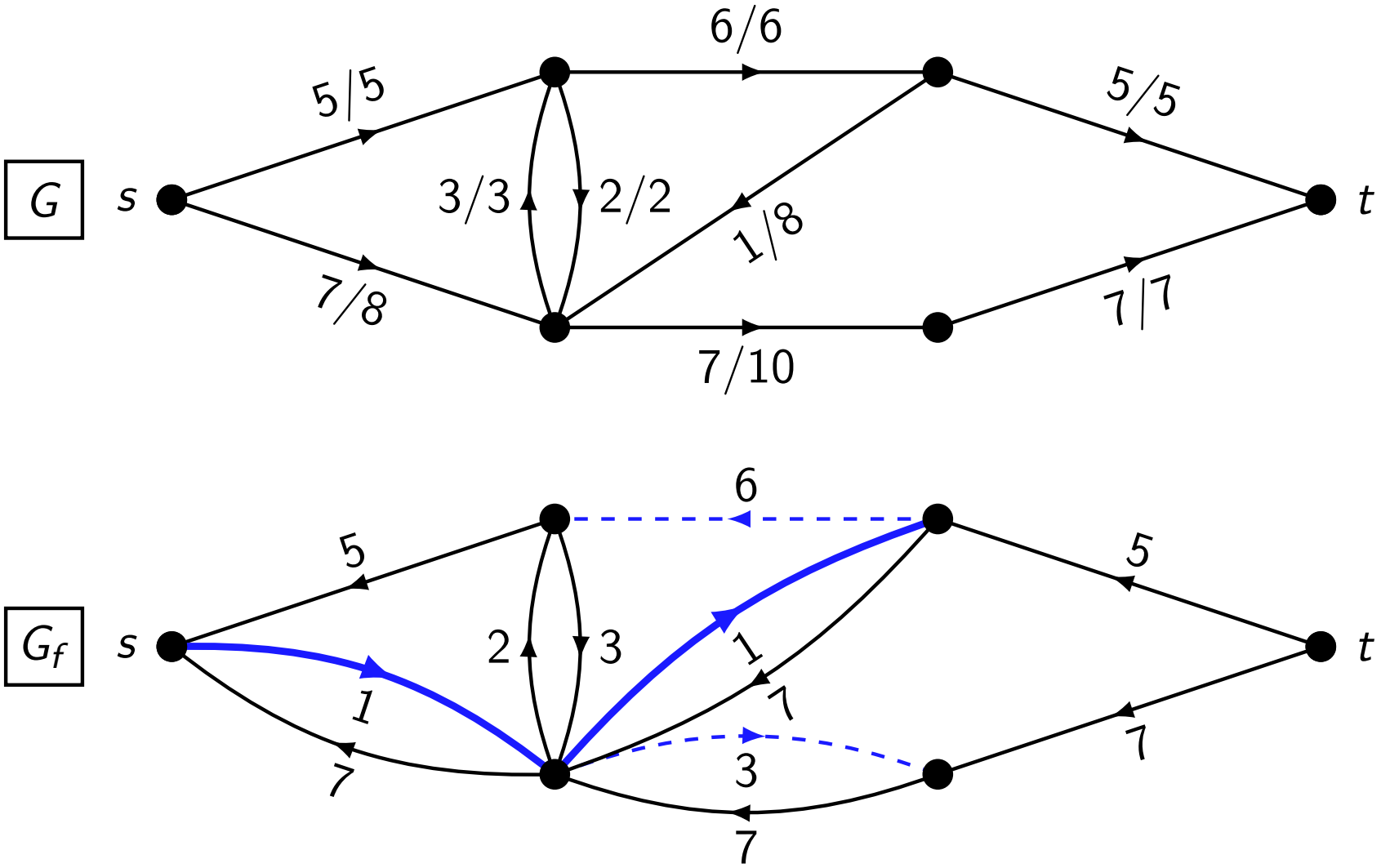

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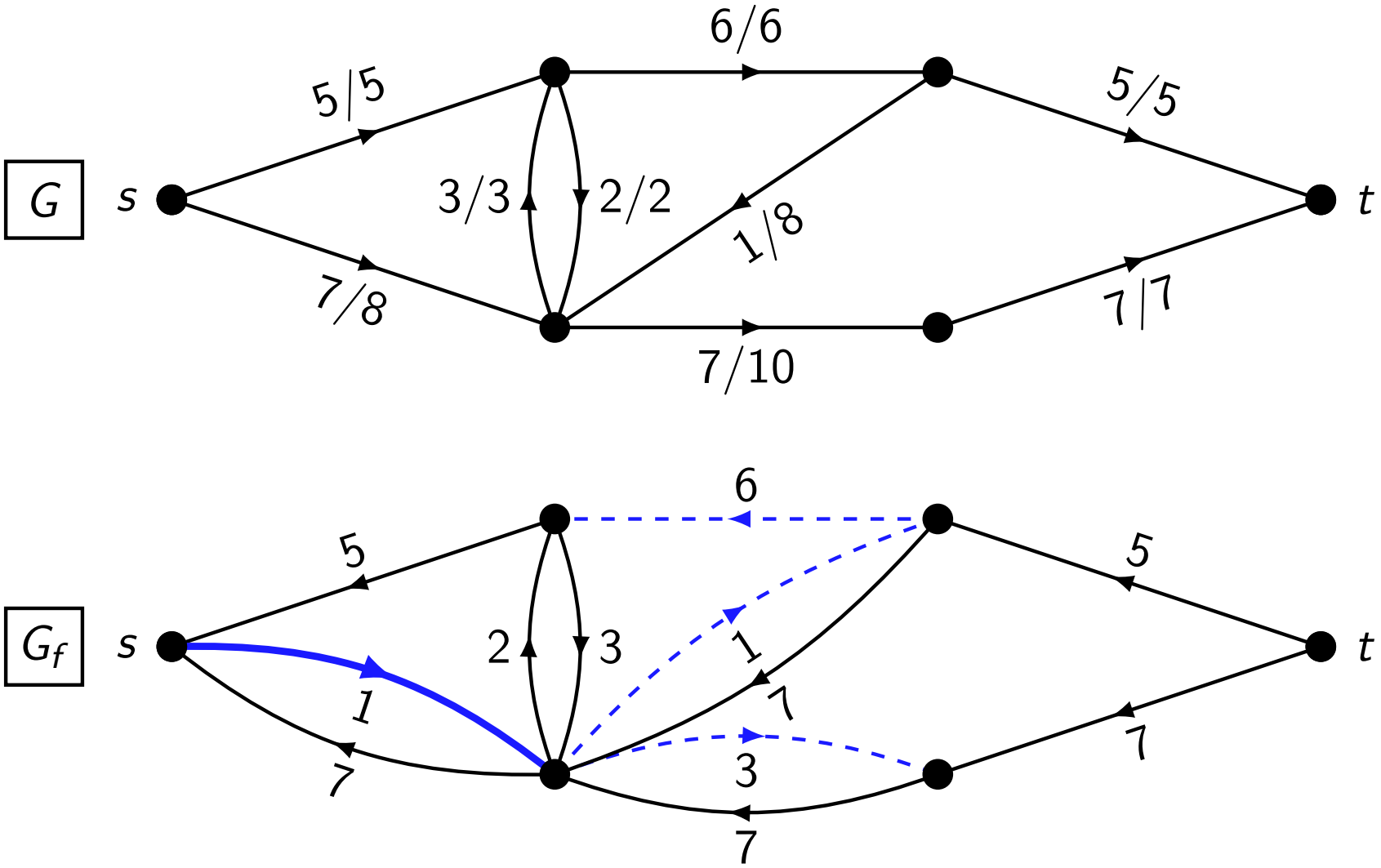

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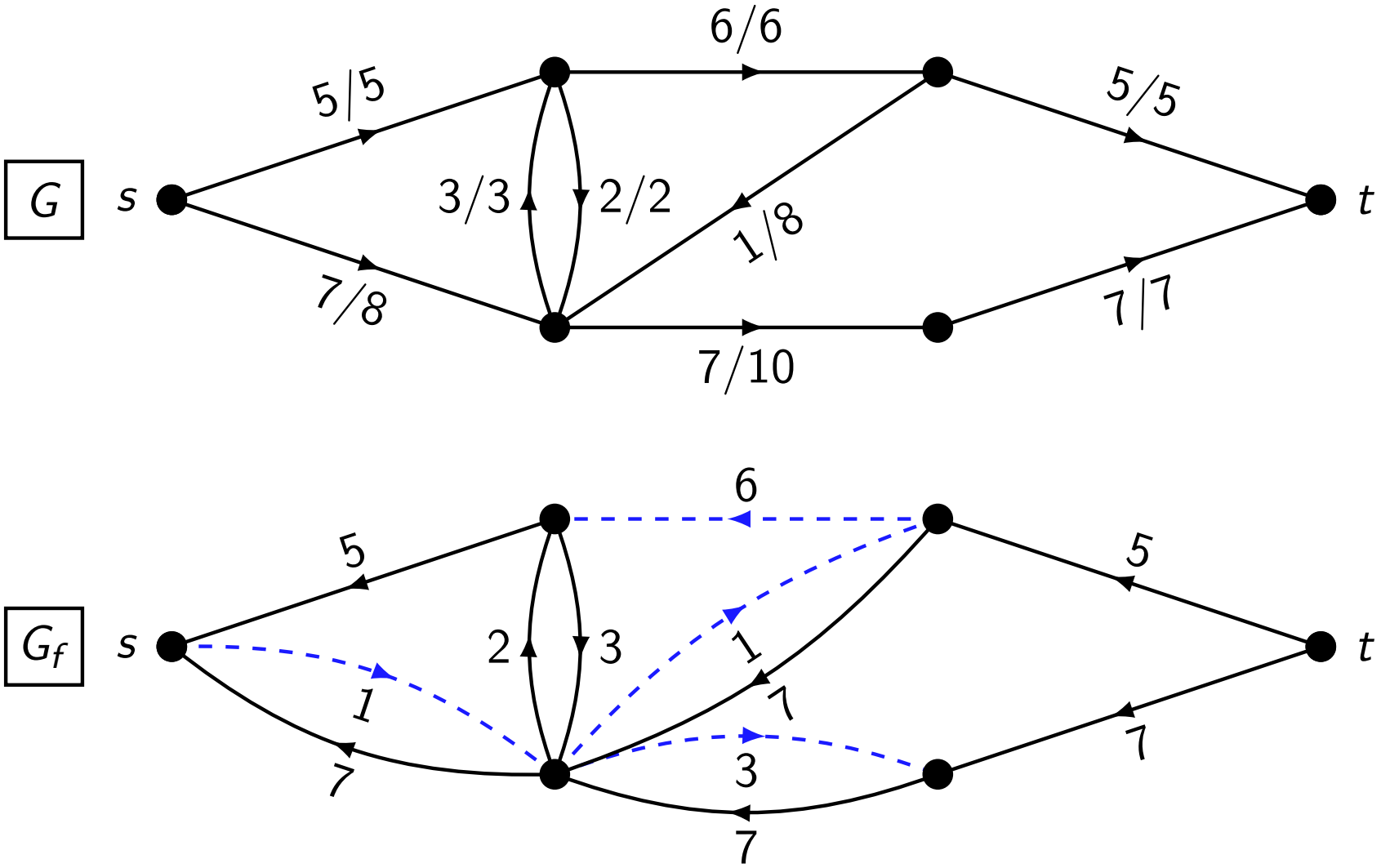

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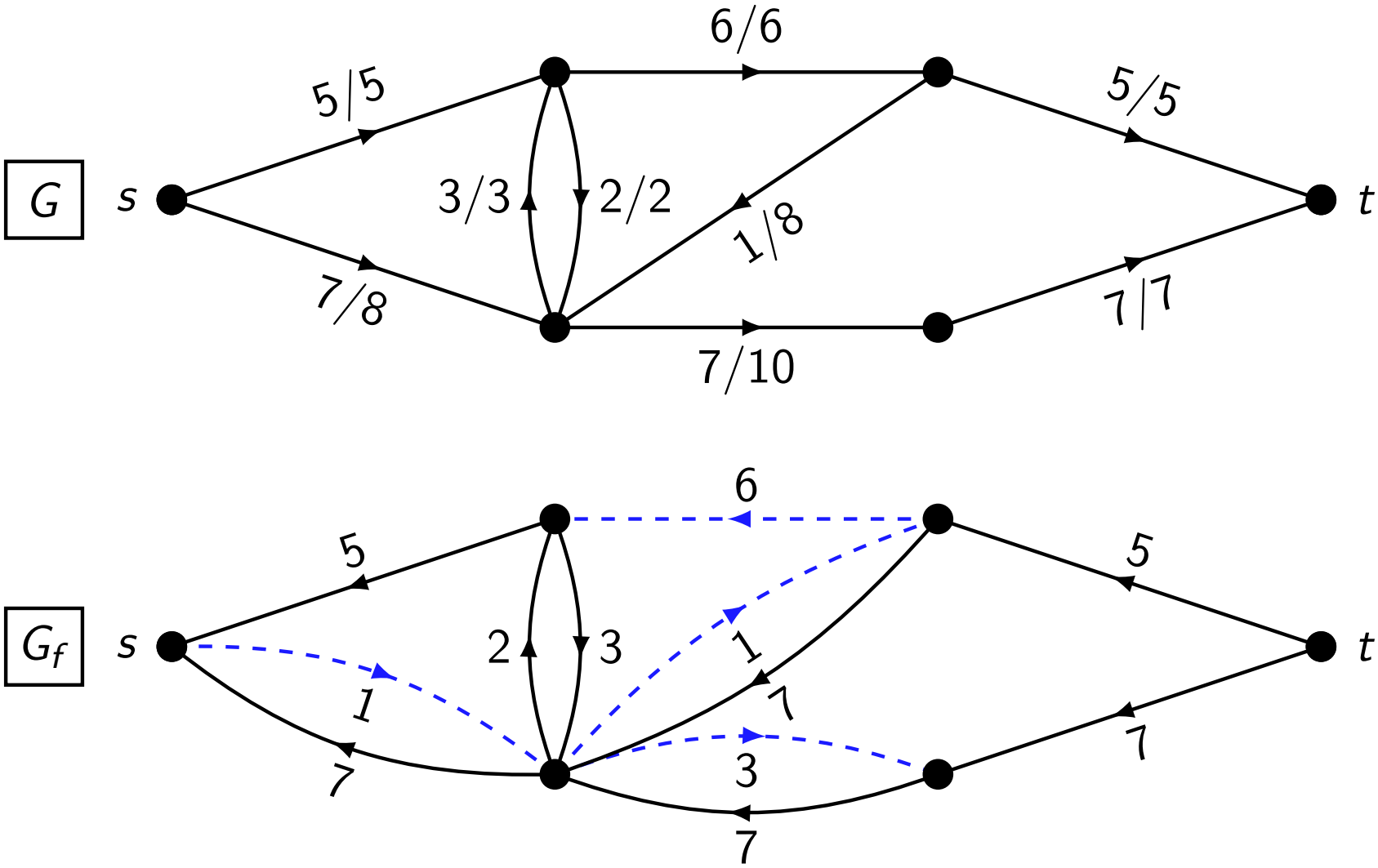
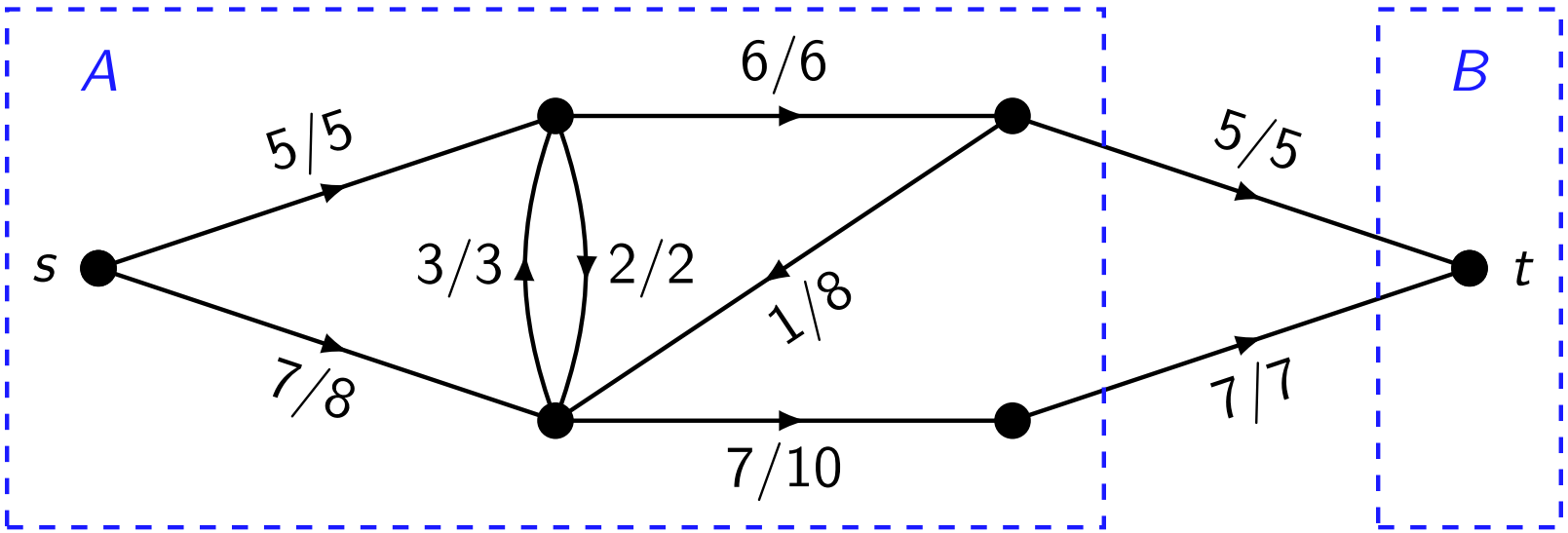


No such path exists, so we're done! This flow has value $5 + 7 = 12$.

Why does this work?

A **cut** is any pair of disjoint sets $A, B \subseteq V$ with $A \cup B = V$, $s \in A$ and $t \in B$. (So A and B partition V, the source is in A and the sink is in B.)

Lemma 2: For all cuts (A, B), $v(f) = f^+(A) - f^-(A) = f^-(B) - f^+(B)$.



Write $c^+(A) = \sum_{e \text{ out of } A} c(e)$. By Lemma 2, **any** flow g has value $v(g) = g^+(A) - g^-(A) \leq c^+(A) = 12$, and our output flow has value 12. So it must be maximum.

We can use the same argument to prove Ford-Fulkerson always works.

Lemma 2: For all cuts (A, B), $v(f) = f^+(A) - f^-(A) = f^-(B) - f^+(B)$.

To prove the flow f returned by Ford-Fulkerson is **always** maximum by this argument, we will show there is **always** a cut (A, B) with $v(f) = c^+(A)$, i.e. with $f^+(A) = c^+(A)$ and $f^-(A) = 0$.

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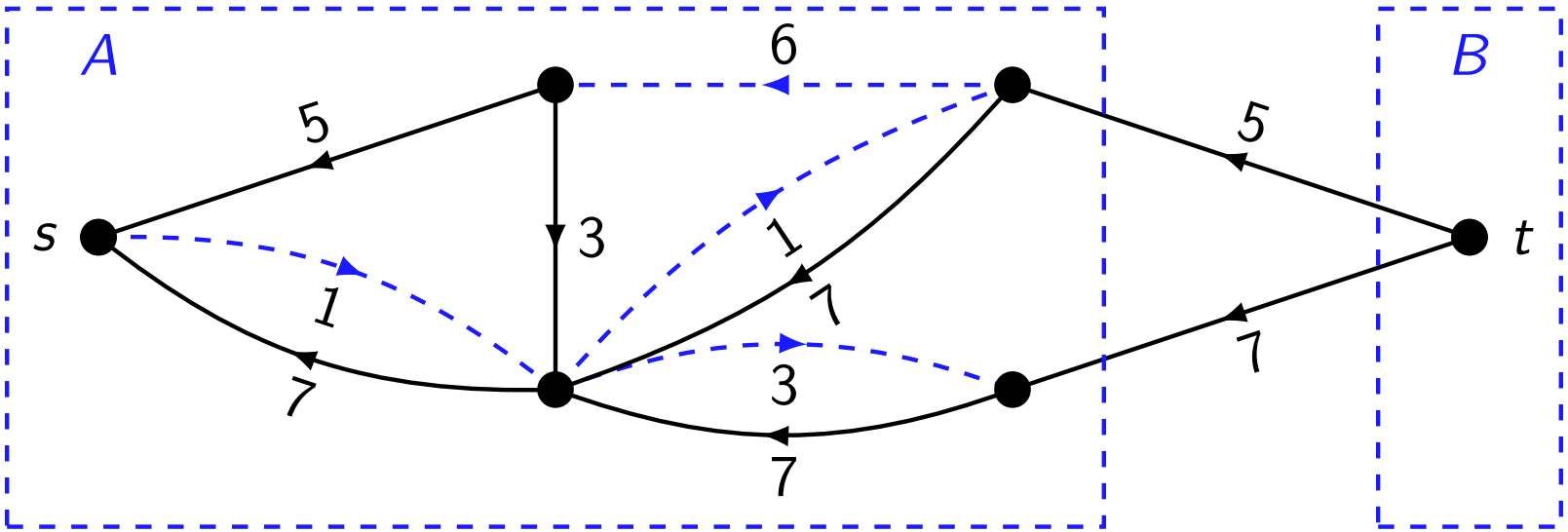

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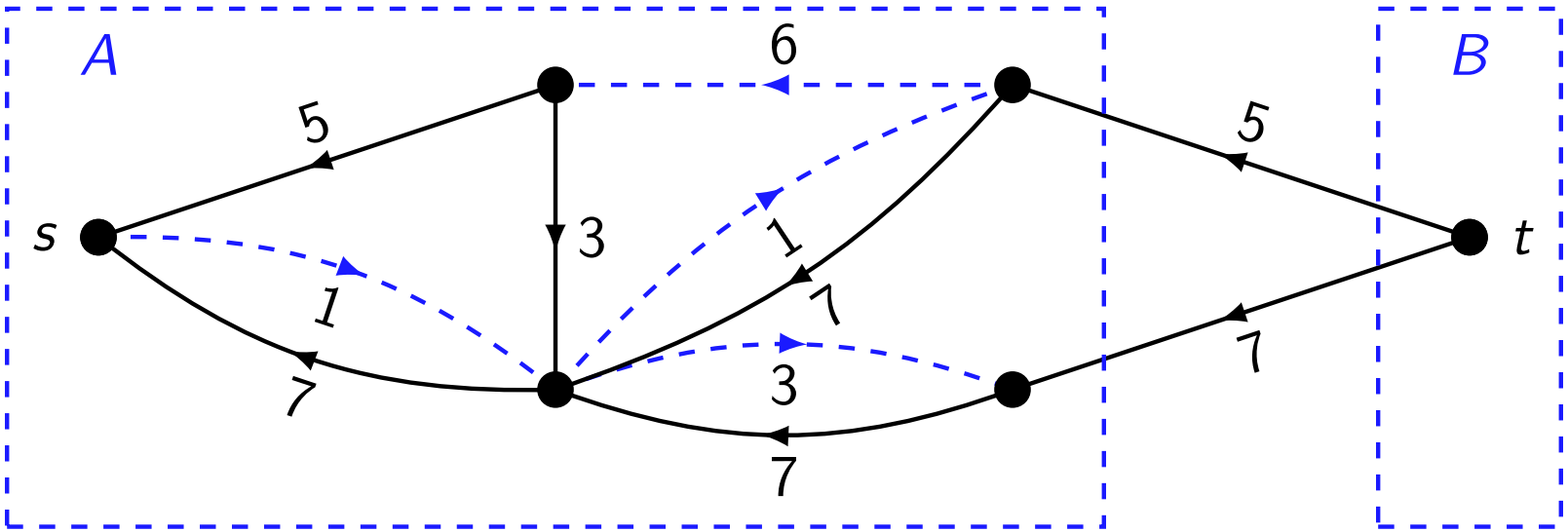

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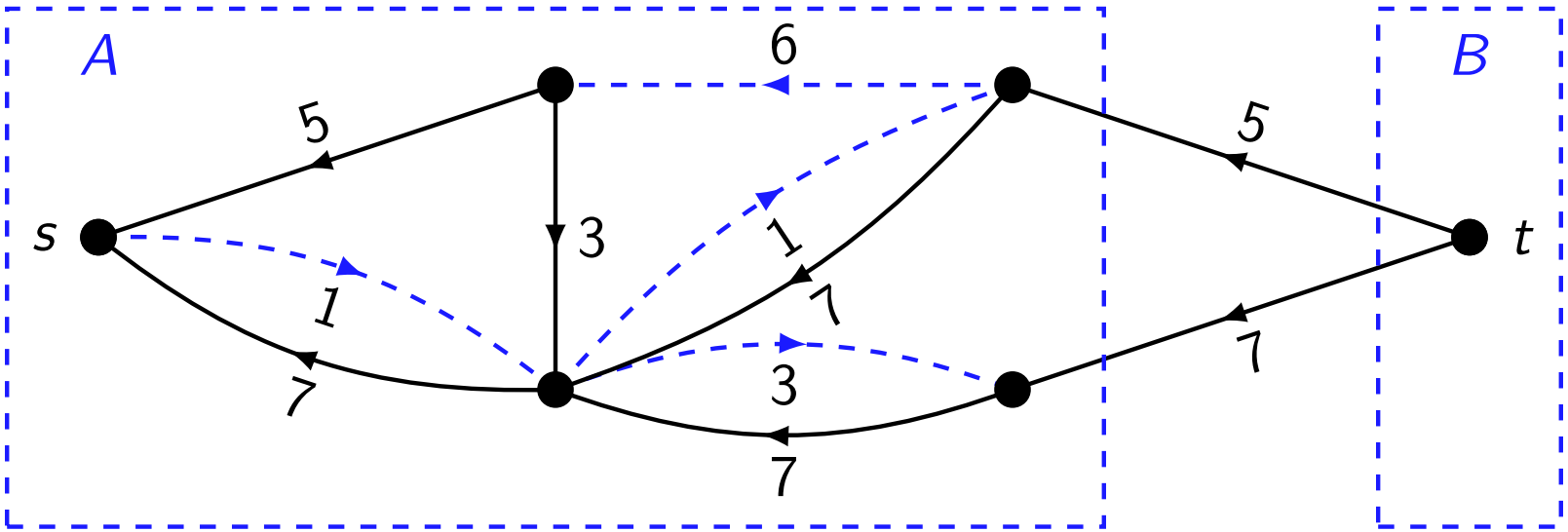

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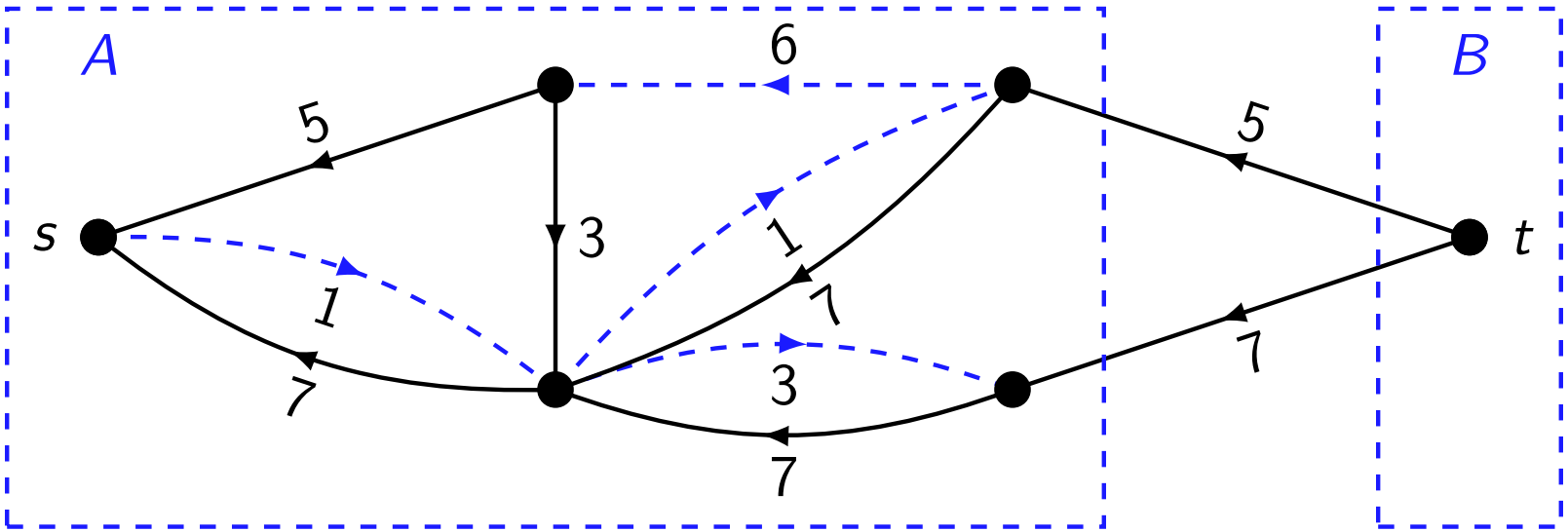

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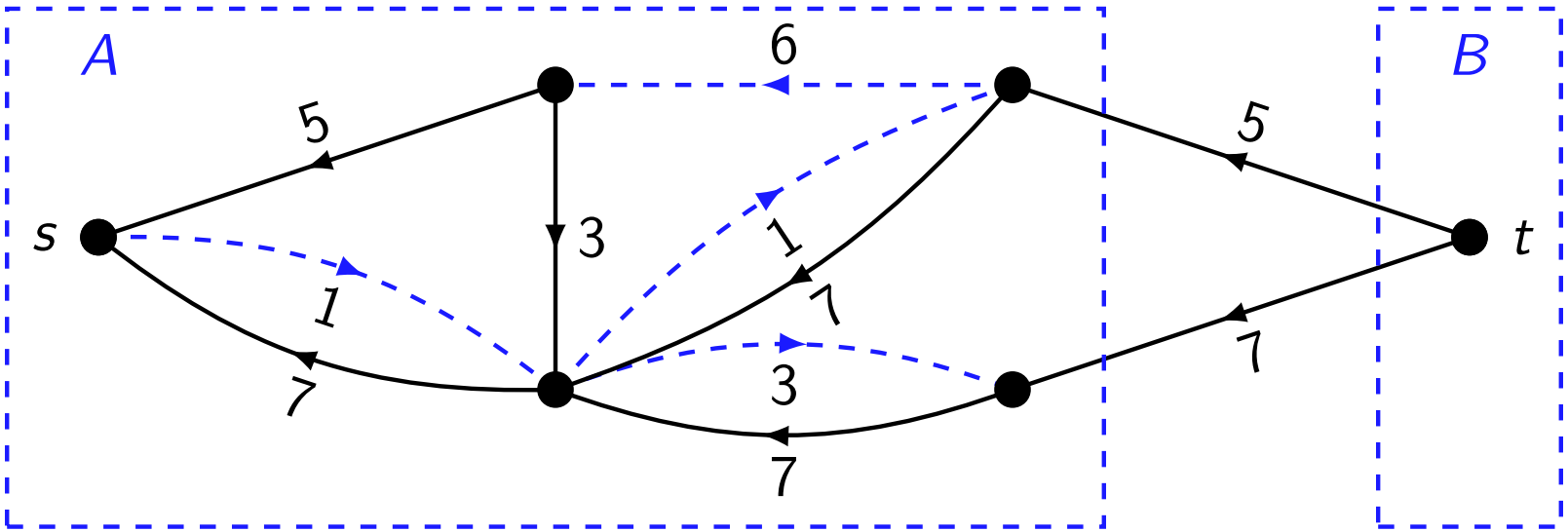

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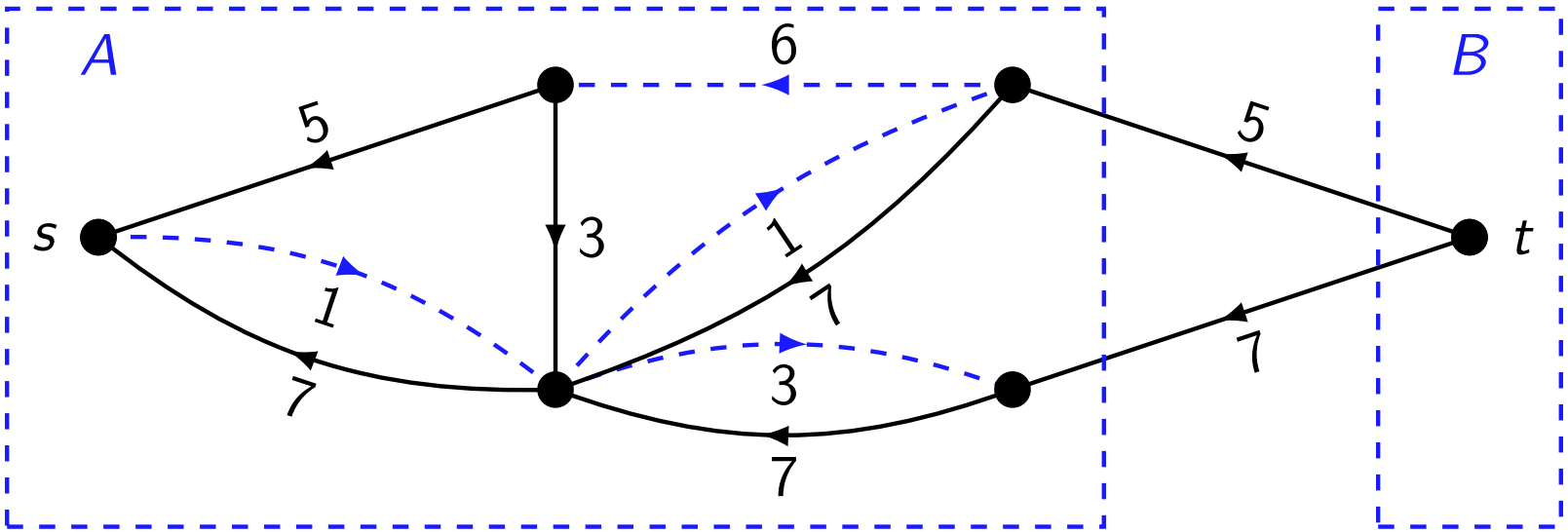

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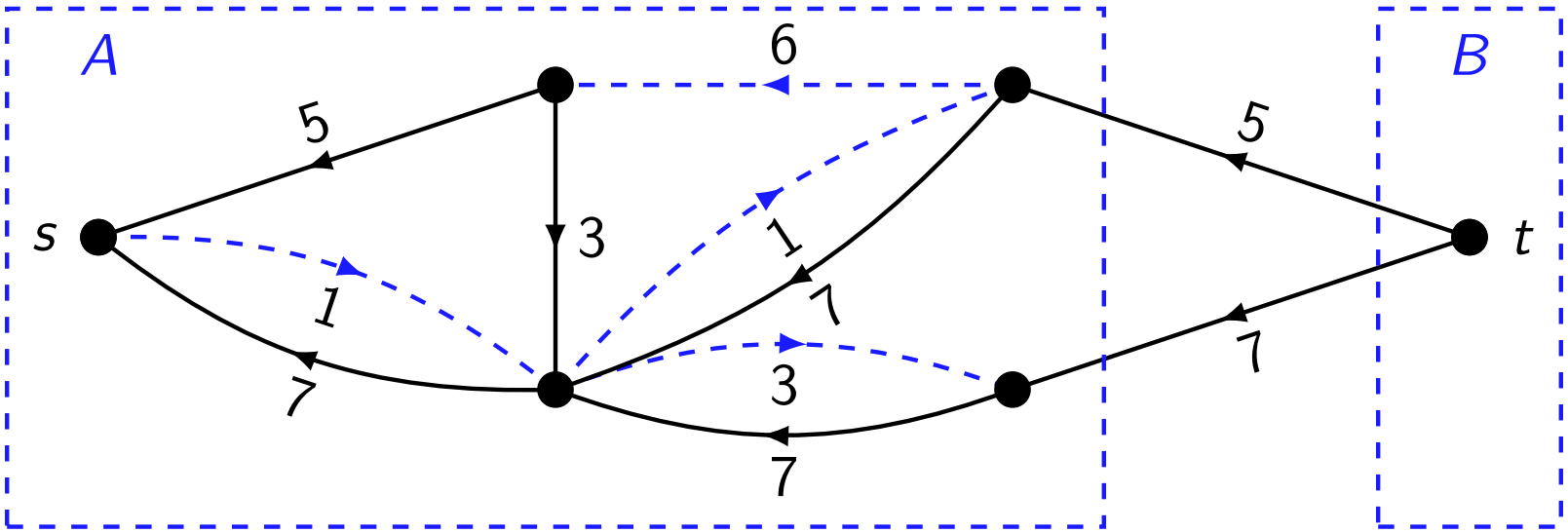

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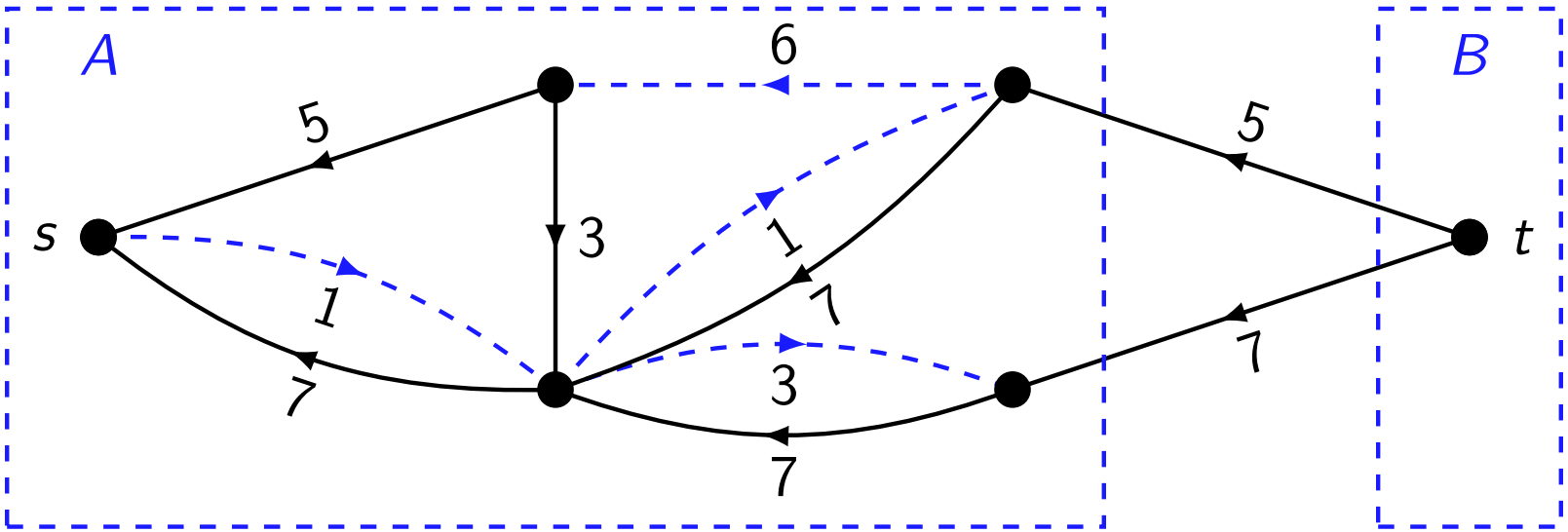

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So by Lemma 2, every other flow g has value $g^+(A) - g^-(A) \leq c^+(A) = v(f)$. Thus f is a maximum flow and Ford-Fulkerson is correct. □

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Proof: Let f be a maximum flow, and let (A, B) be a cut minimising $c^+(A)$. We already proved $v(f) \leq c^+(A)$. Moreover, there is no augmenting path for f, so exactly as before, there is a cut (A', B') with $c^+(A') = v(f)$; thus $v(f) \geq c^+(A)$. The result follows. □