Independent sets and vertex covers
COMS20010 (Algorithms II)

John Lapinskas, University of Bristol

Examples of NP-hardness

We can prove a problem is NP-hard by reducing from 3-SAT... but we can also do it by reducing from any other NP-complete problem.

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- Donkey Kong Country 1–3;
- Every Legend of Zelda game;
- Every Metroid game;
- Every Fire Emblem game;
- Mainline Pokémon games;
- Mario Kart;
- Desktop Tower Defense;
- Harvest Moon;
- Inventory packing in ARPGs;
- Damage boosting in speedruns.

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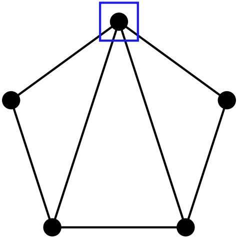

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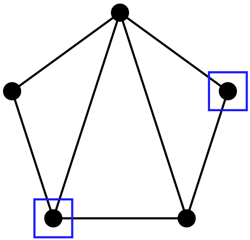

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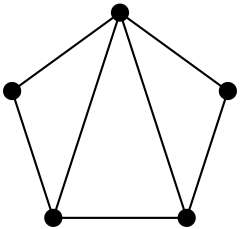

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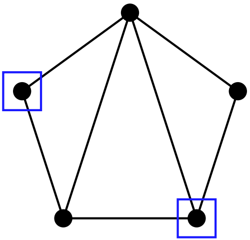

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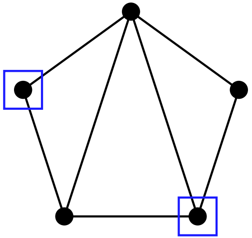

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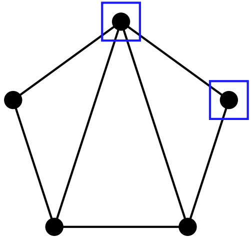

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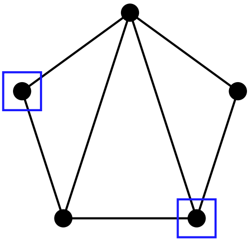

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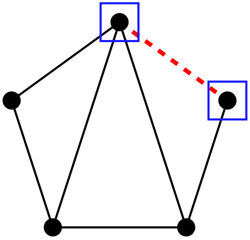

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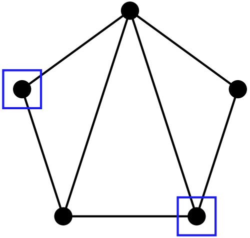

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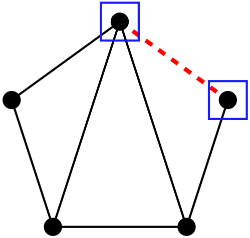

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Independent sets are important in graphs which model **conflicts**.

For example, suppose we are trying to assign frequencies to radio transmitters while avoiding interference. If we join two transmitters by an edge when they are close enough to interfere with each other, then we can safely assign the same frequency to all transmitters in an independent set.

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The decision version of this problem, **IS**, asks: Given a graph $G = (V, E)$, and an integer k, does G contain an independent set of size at least k?

Theorem: IS is NP-complete.

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We will show NP-hardness by reducing from 3-SAT, i.e. proving 3-SAT $\leq_c$ IS. Since we already proved SAT $\leq_c$ 3-SAT, the result follows.

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A CNF formula has width 3 if all its OR clauses contain 3 literals.
3-SAT asks: is the input width-3 CNF formula satisfiable?

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Let F be an instance of 3-SAT. We'll follow our usual approach: build a graph G whose size-($\geq k$) independent sets correspond to satisfying assignments of F, then apply our IS oracle to G.

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An independent set can't contain both vertices, and (if we do everything else right) a **maximum** independent set must contain one of the two vertices.

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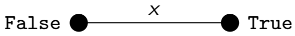

We use the same idea to model the clauses of F. We have three literals in the clause, and we want to force one of them to be true, so...

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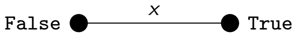
$x \longrightarrow$ 

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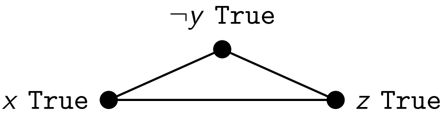
$x \vee \neg y \vee z \longrightarrow$ 

We will set things up so that:
Maximum independent set $\implies$ exactly one vertex is included.

Joining the gadgets together

So if $F = (x \vee \neg y \vee z) \wedge (w \vee \neg x \vee \neg z)$, say, how do we build G?

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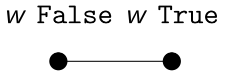

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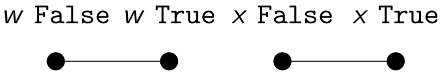

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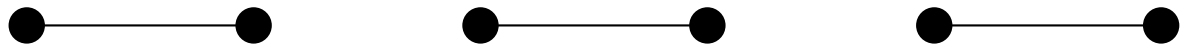

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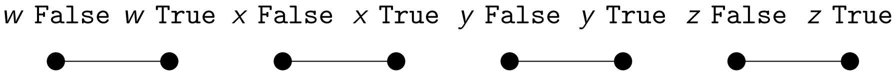

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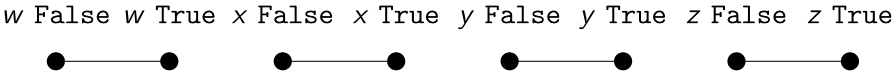

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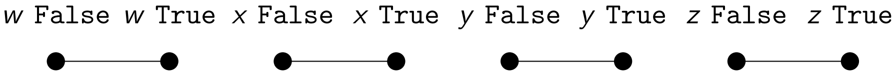

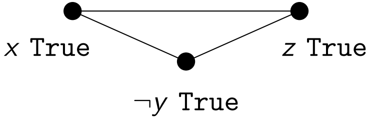

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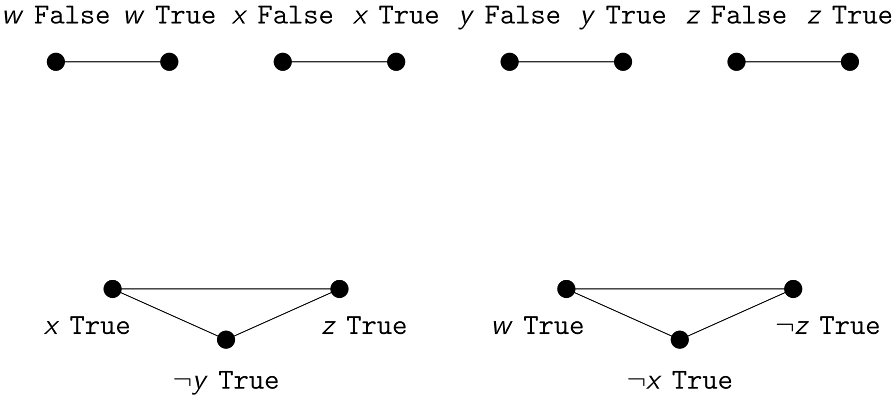

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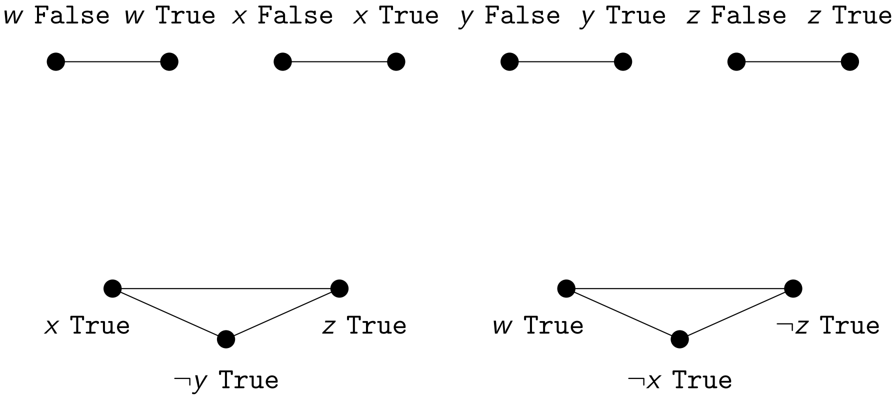

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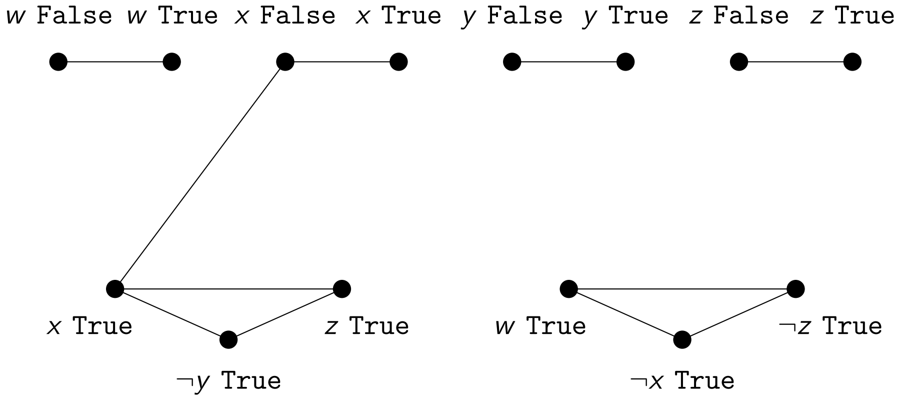

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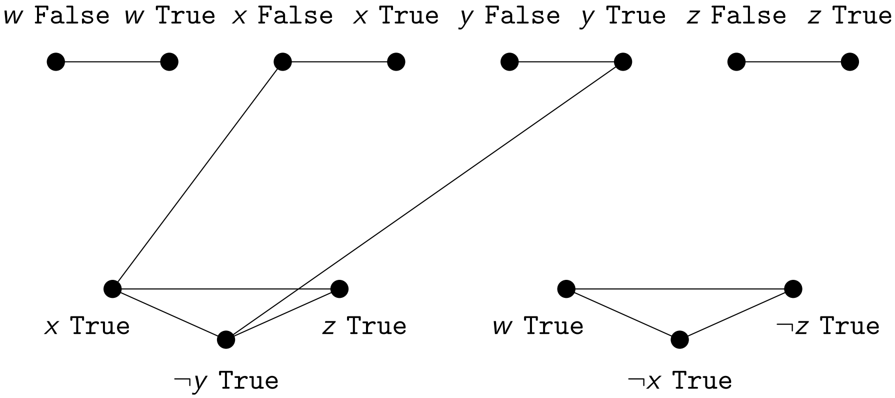

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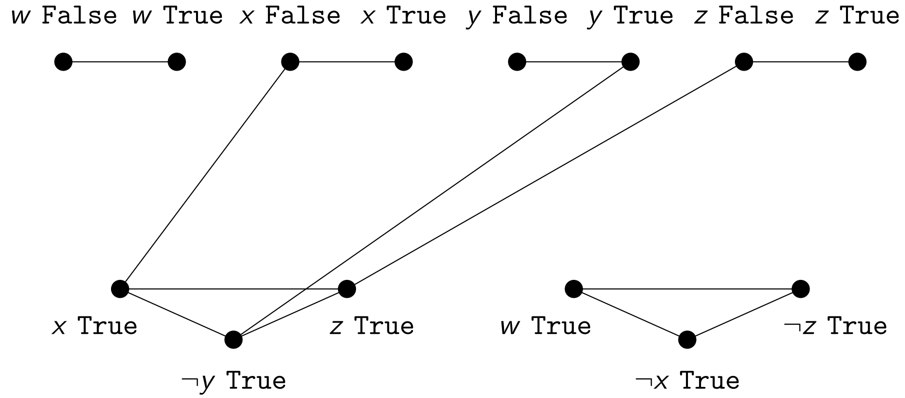

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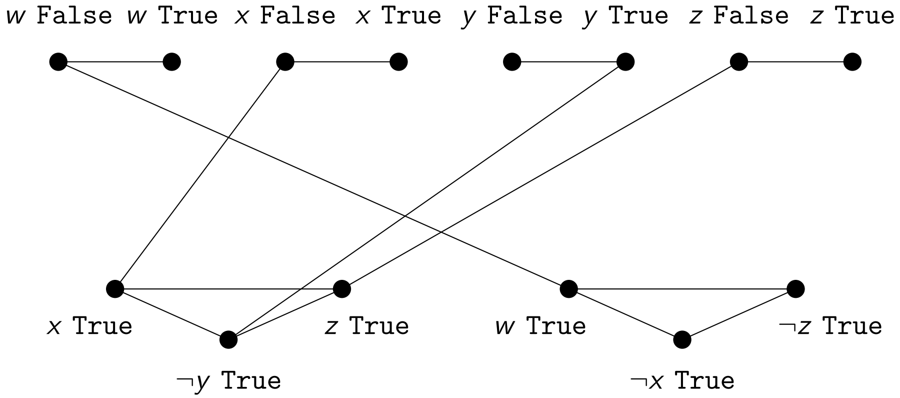

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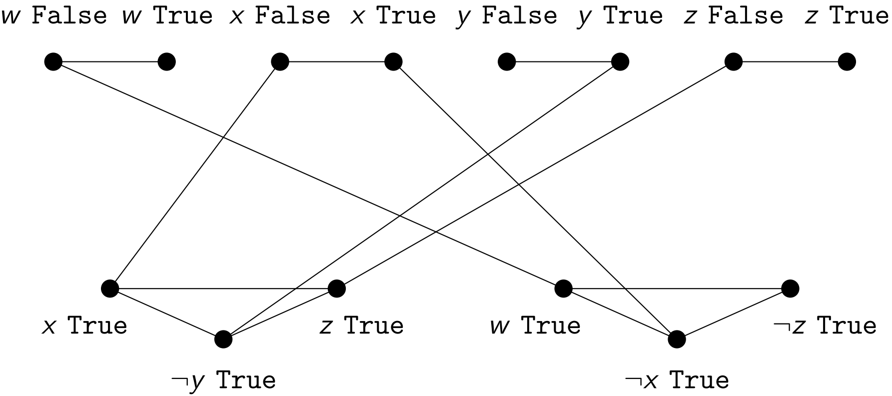

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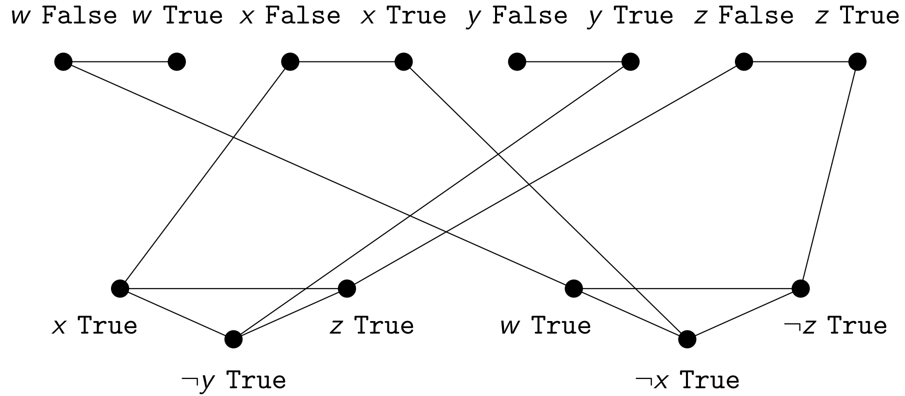

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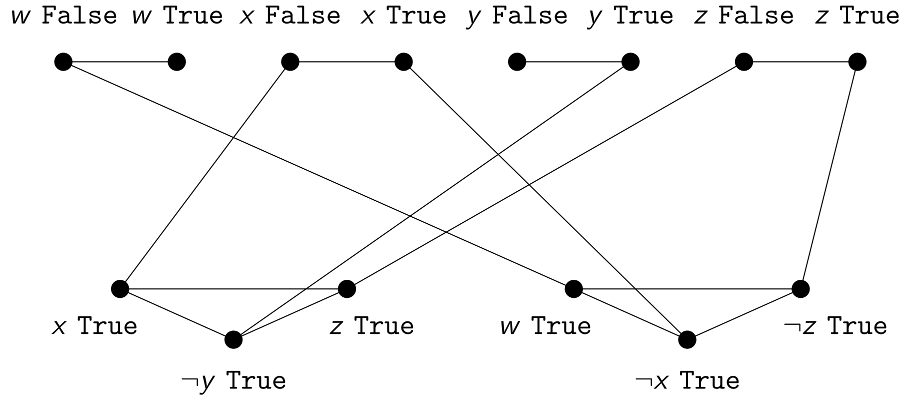

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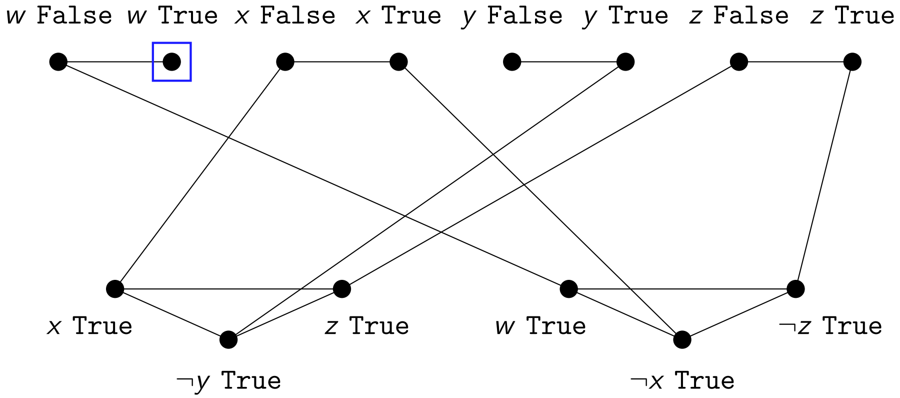

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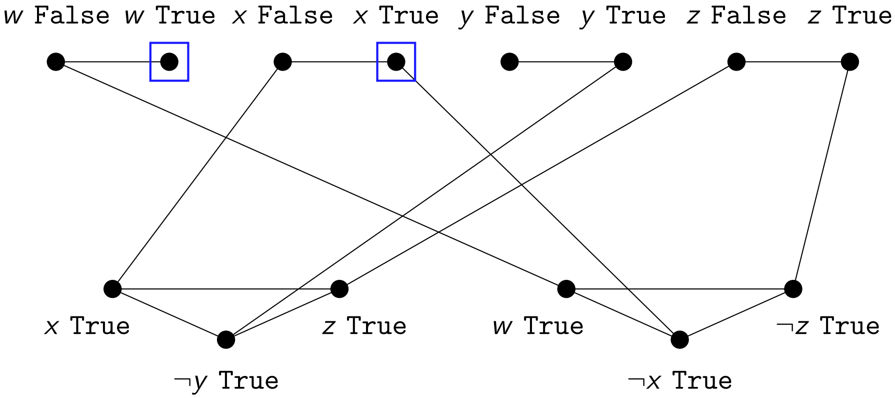

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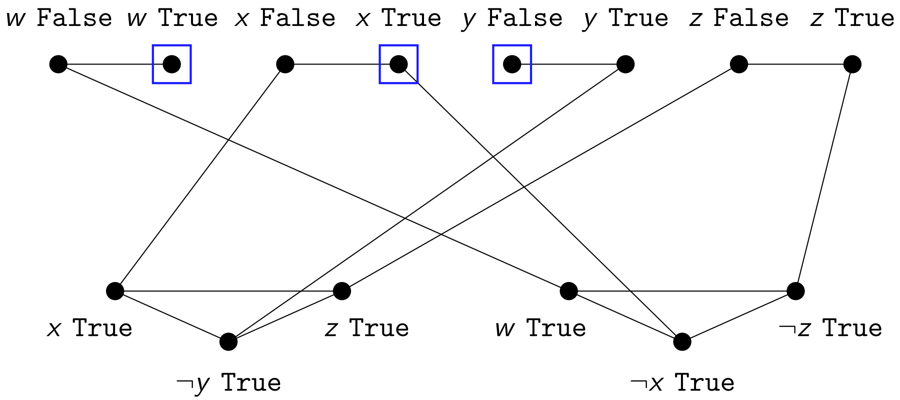

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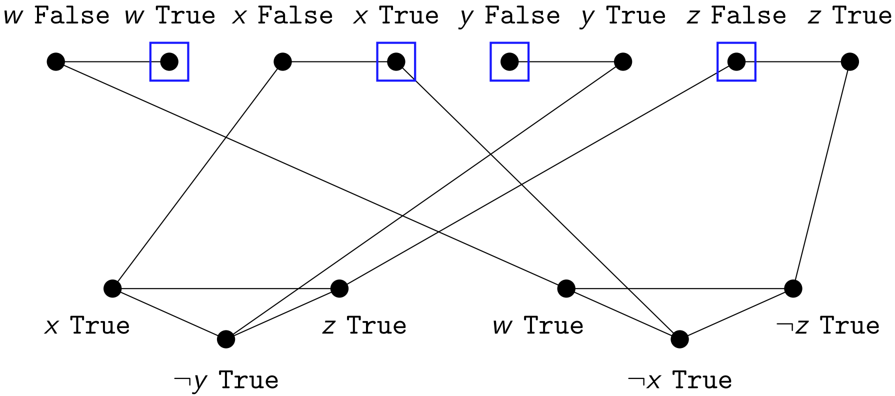

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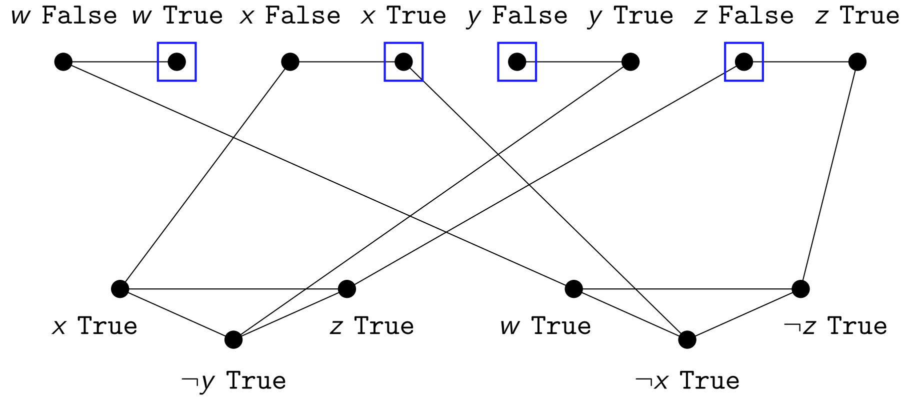

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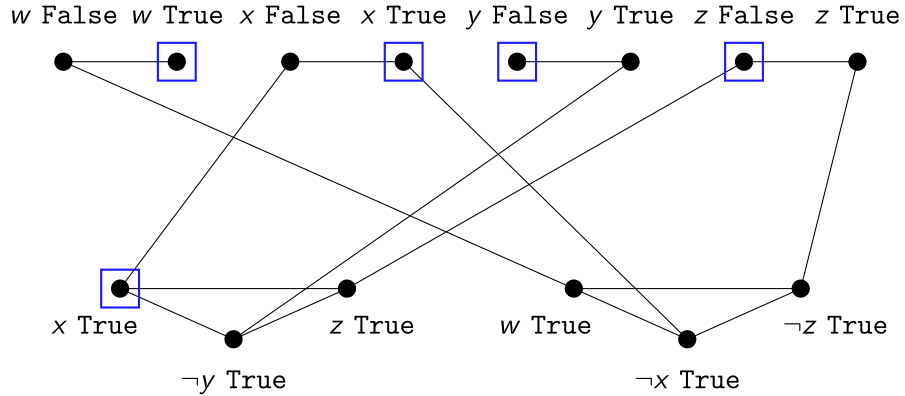

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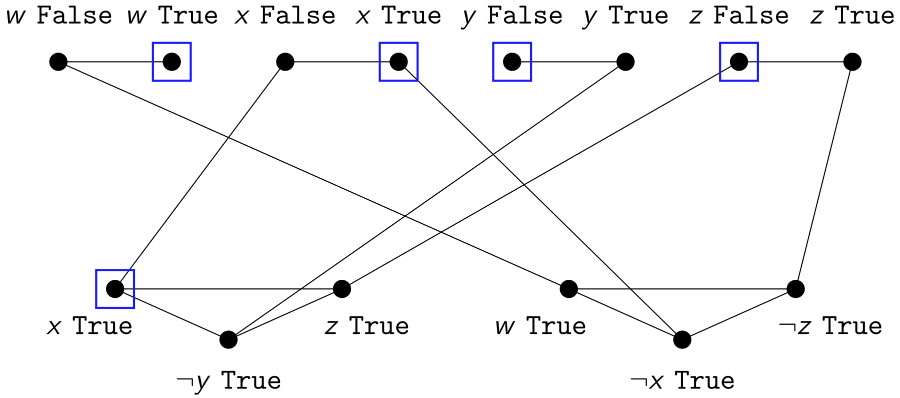

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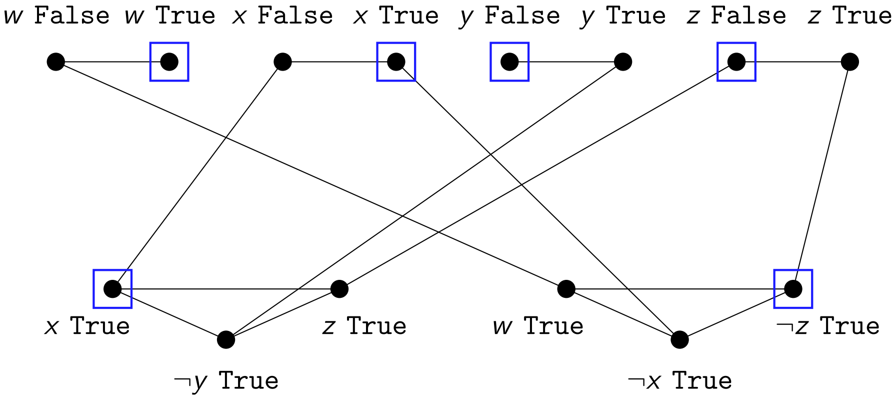

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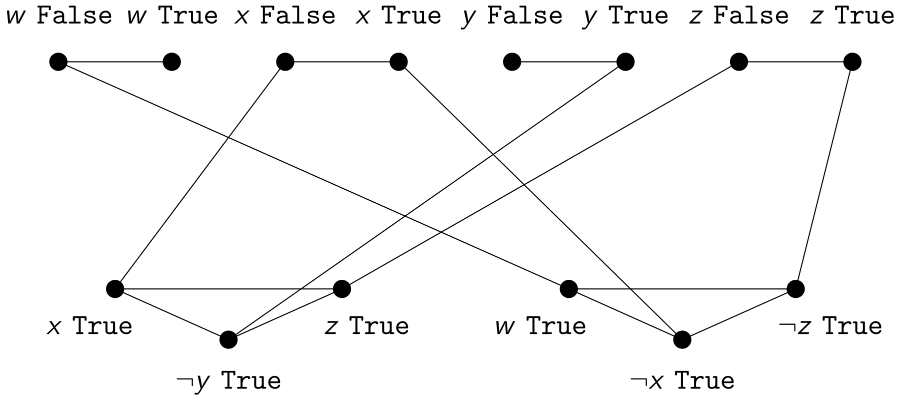

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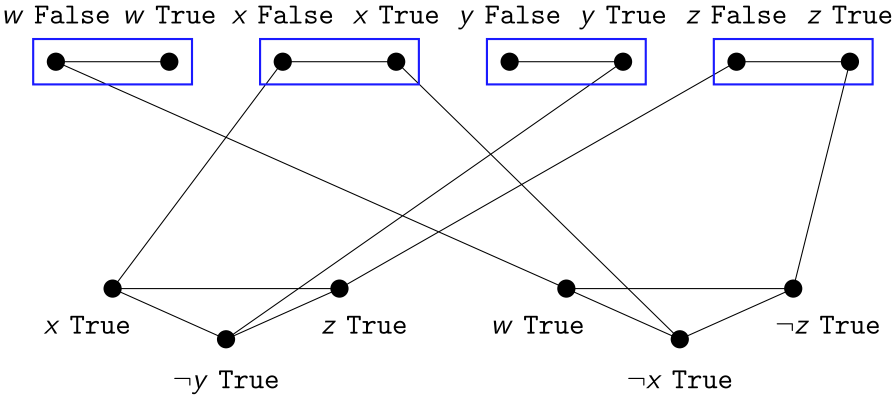

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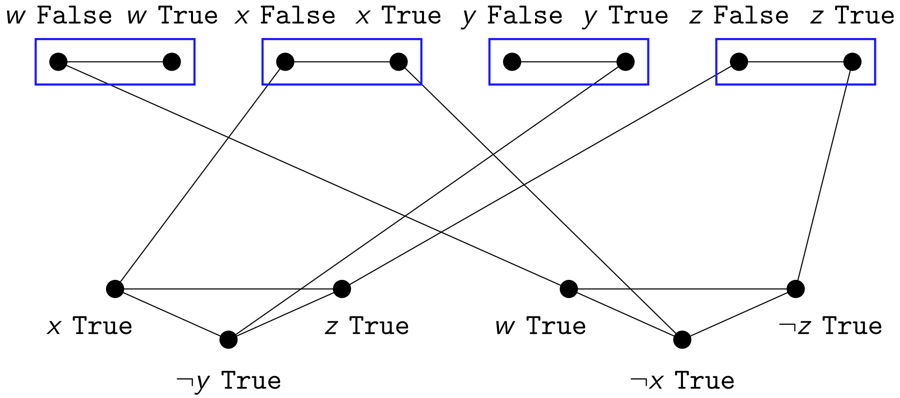

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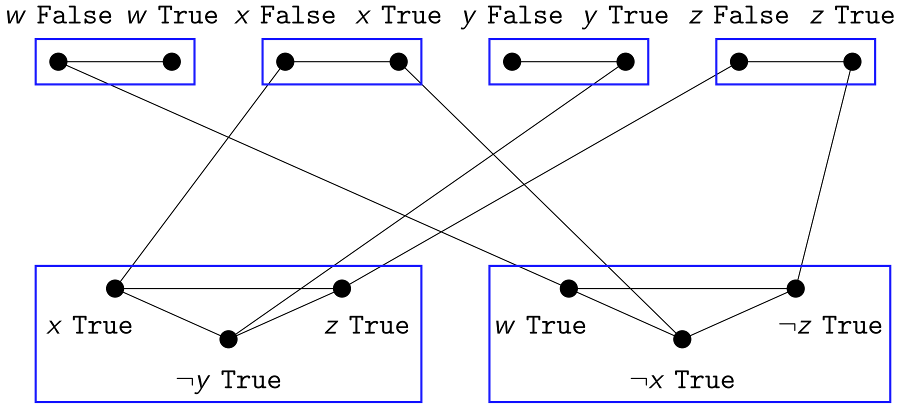

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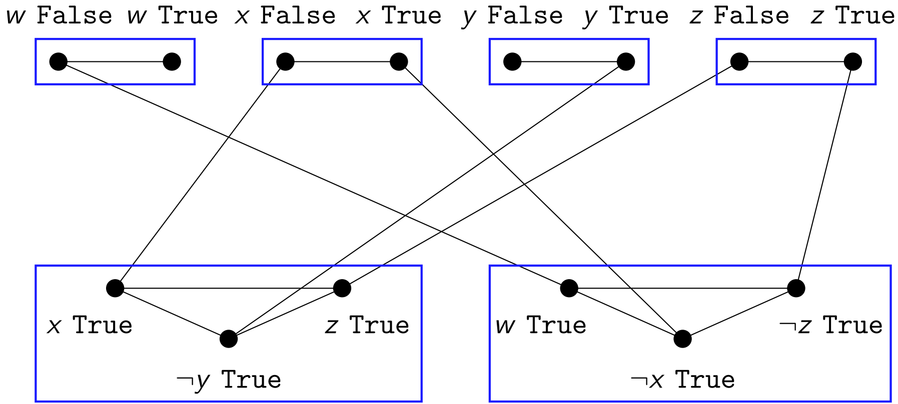

So any size-($\geq$6) independent set must have size **exactly** 6, and contain one vertex from each variable gadget and one from each clause gadget.

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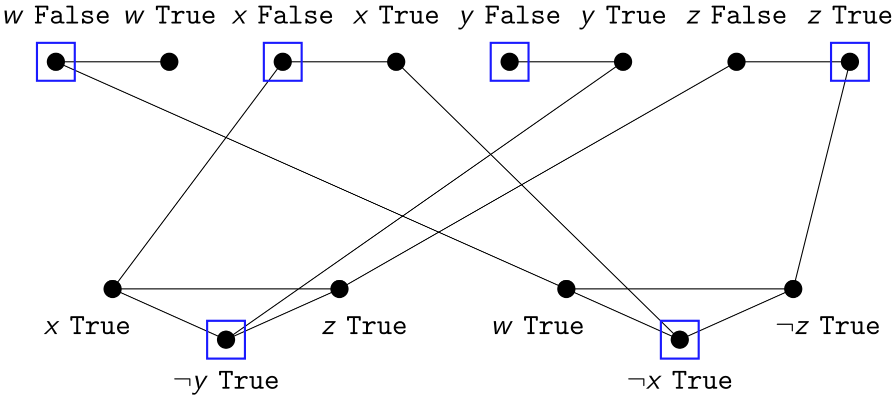

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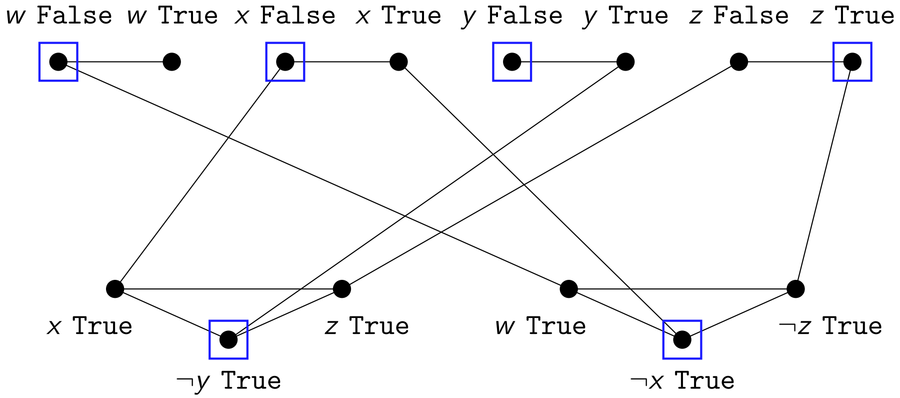

Hence any size-($\geq$6) independent set corresponds to an assignment, in this case $w = x = y =$ False, $z =$ True. It must be satisfying because there are no edges between variable vertices and clause vertices.

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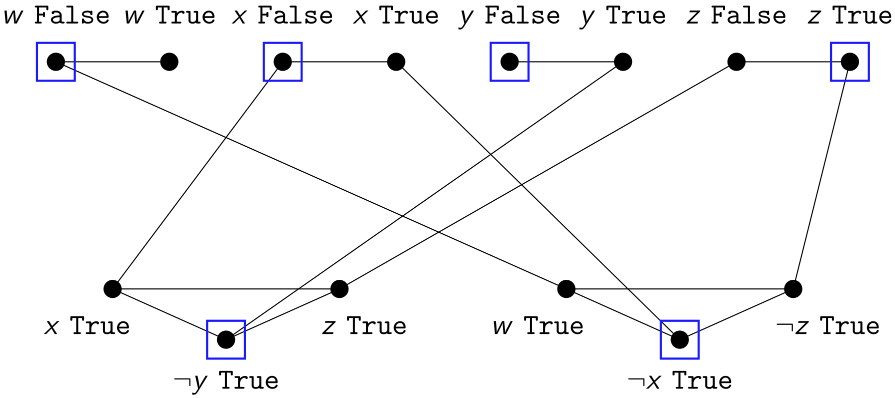
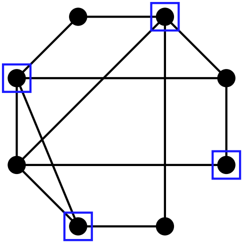


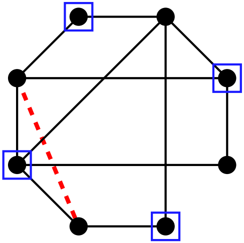
The same construction (and the same correctness proof) works for any instance of 3-SAT. □

Recall from Video 8-2...

A **vertex cover** in a graph $G = (V, E)$ is a set $X \subseteq V$ such that every edge in E has at least one vertex in X.



A valid vertex cover.



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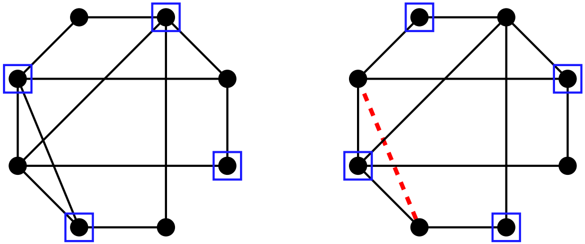

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We claimed this problem was hard to solve exactly (our algorithm was approximate), but we never proved it...

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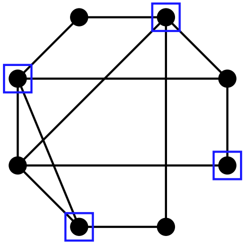

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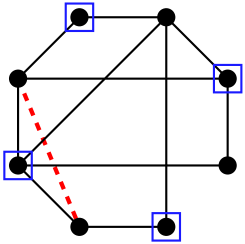

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The decision version of this problem (**VC**) asks: Given a graph G, and an integer k, does G contain a vertex cover of size at most k?

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Our reduction just passes the instance $(G, |V| - k)$ to our VC-oracle. $\square$

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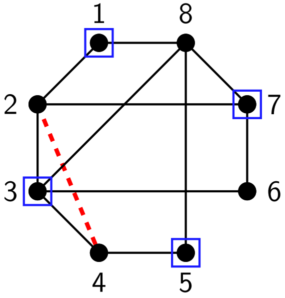

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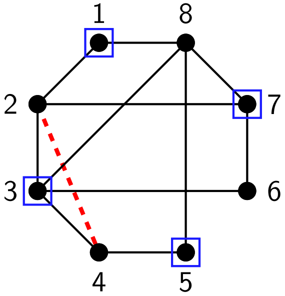

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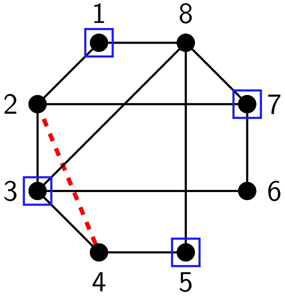

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Notice we reduced SAT $\leq_c$ 3-SAT $\leq_c$ IS $\leq_c$ VC $\leq_c$ ILP — by proving one problem is NP-hard, we make all our future hardness proofs easier...