NP versus Co-NP
COMS20010 (Algorithms II)

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The asymmetry of NP

Given a decision problem X, we write $\overline{X}$ for its **complement**:

Yes instances of X are No instances of $\overline{X}$ and vice versa.

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Conjecture: In general, this isn't possible. If it were, it wouldn't mean much for algorithms, but it would be a revolution in mathematics.

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Cook reductions are useless here, since every problem in NP reduces to $\overline{\mathrm{SAT}}$ and every problem in Co-NP reduces to SAT.

We need a notion of reduction that can make finer distinctions and tell the two complexity classes apart...

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Karp reductions are **stronger** than Cook reductions; $X \leq_K Y \Rightarrow X \leq_c Y$, since we can apply our oracle to $f(x)$, but the reverse doesn't hold.

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As with Cook reductions, we say a decision problem Y is **NP-hard under Karp reductions** if $X \leq_K Y$ for all $X \in \text{NP}$.

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The proof of Cook-Levin implies SAT is NP-complete under Karp reductions, and $\overline{\mathrm{SAT}}$ is Co-NP-complete under Karp reductions.

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For SAT $\leq_C$ 3-SAT, we built a 3-SAT instance with the same answer as the SAT instance...

$$F = u \wedge (\neg u \vee \neg v) \wedge (v \vee \neg w \vee x \vee \neg y \vee \neg z) \wedge (y \vee z) \wedge (\neg v \vee w \vee z)$$

$$\downarrow$$

$$\begin{aligned} F' = {} & (u \vee f_1 \vee f_2) \wedge (\neg u \vee \neg v \vee f_1) \wedge (e_1 \vee \neg v \vee f_1) \wedge (e_1 \vee w \vee f_1) \wedge (\neg e_1 \vee v \vee \neg w) \\ & \wedge (e_2 \vee \neg e_1 \vee f_1) \wedge (e_2 \vee \neg x) \wedge (\neg e_2 \vee e_1 \vee x) \wedge (e_3 \vee \neg e_2 \vee f_1) \wedge (e_3 \vee y \vee f_1) \vee (\neg e_3 \vee e_2 \vee \neg y) \\ & \wedge (e_3 \vee \neg z \vee f_1) \wedge (y \vee z \vee f_1) \wedge (\neg v \vee w \vee z) \wedge (\neg f_1 \vee a_1 \vee a_2) \wedge (\neg f_1 \vee a_1 \vee \neg a_2) \wedge (\neg f_1 \vee \neg a_1 \vee a_2) \\ & \wedge (\neg f_1 \vee \neg a_1 \vee \neg a_2) \wedge (\neg f_2 \vee a_1 \vee a_2) \wedge (\neg f_2 \vee a_1 \vee \neg a_2) \wedge (\neg f_2 \vee \neg a_1 \vee a_2) \wedge (\neg f_2 \vee \neg a_1 \vee \neg a_2). \end{aligned}$$

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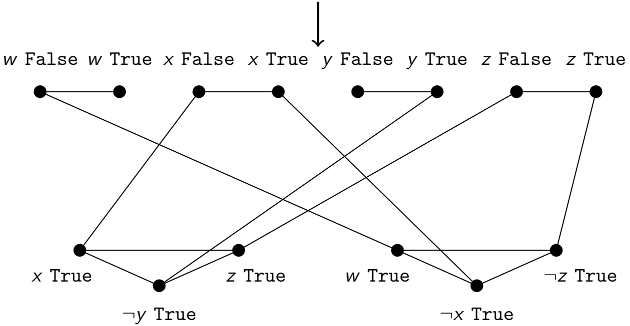
For 3-SAT $\leq_C$ IS, we built an independent set instance with the same answer as our 3-SAT instance...

$$F = (x \vee \neg y \vee z) \wedge (w \vee \neg x \vee \neg z)$$



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For IS $\leq_C$ VC, we built a vertex cover instance with the same answer as our independent set instance...

$$(G, k) \longrightarrow (G, |V| - k).$$

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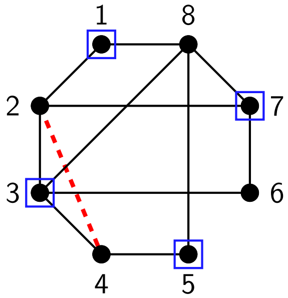
And for VC $\leq_C$ ILP, we built an integer linear programming instance with the same answer as our vertex cover instance.



$\longrightarrow$

$$\begin{aligned} \textstyle\sum_v x_v &\to \min \text{ subject to} \\ x_u + x_v &\geq 1 \text{ for all } \{u, v\} \in E; \\ x_v &\leq 1 \text{ for all } v \in V; \\ x_v &\geq 0 \text{ for all } v \in V; \\ x_v &\in \mathbb{N} \text{ for all } v \in V. \end{aligned}$$

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And we believe there **are** problems which are NP-complete under Cook reductions but not under Karp reductions.

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The worse news: Different people use different definitions. Complexity theorists use Karp reductions, programmers use Cook reductions. And both groups usually just say "NP-hard" or "NP-complete".

In this course: If I don't give more detail, "NP-complete" means under Karp reductions, and "NP-hard" means under Cook reductions.

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The slightly better news: *Almost* every NP-complete problem is NP-complete under both Cook and Karp reductions. So thinking only in terms of Karp reductions still saves effort without really sacrificing power.

You can skip trying to come up with an algorithm and move straight to coming up with gadgets, trying to simulate aspects of one problem using the other.

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And in areas where Karp-unfriendly reduction techniques are more common (e.g. counting problems), everyone just uses Cook reductions, even the pure theorists.