Dealing with NP-hard problems
COMS20010 (Algorithms II)

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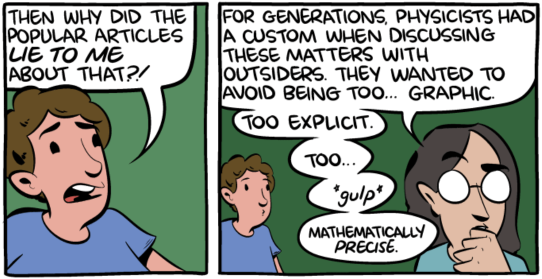

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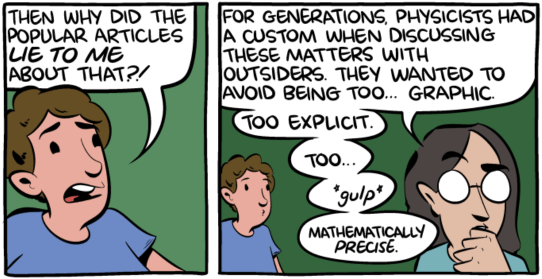

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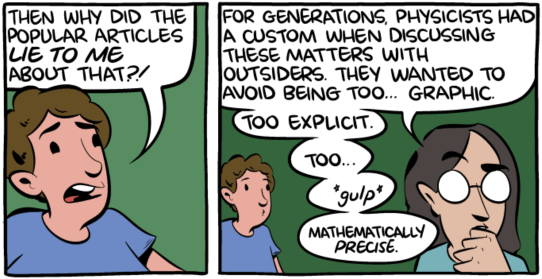

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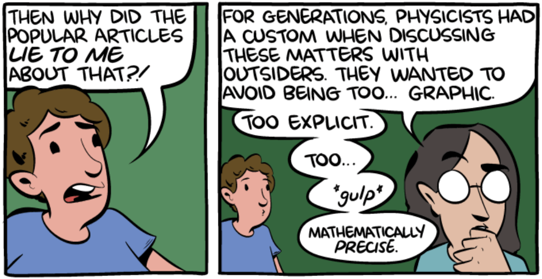

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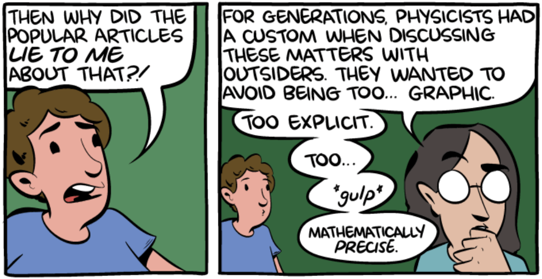

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- Breaking public-key cryptography with e.g. fast prime factorisation;
- Simulating quantum systems.

That's still very useful for e.g. drug discovery! But it's not solving SAT. There's no reason to think your home PC will ever need a "quantum chip".

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Then your output will be a quantum state which, when you measure it, gives `No` with probability $1 - 2^{-n}$ and `Yes` with probability 2^{-n}.
And measuring the state destroys it.

There's **provably** no way to distinguish this state from a pure "`No`" with any reasonable probability.

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The best evidence for quantum computers being more powerful than classical computers is that they can factor large numbers quickly.

But the evidence this is a hard problem is much weaker than e.g. $\mathrm{P} \neq \mathrm{NP}$.
It's in $\mathrm{NP} \cap$ Co-NP, and some people do think this is contained in P...

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And it's NP-hard to approximate independent set even to within a factor of $n^{0.9999}$ on an n-vertex graph. (You can approximate it to within a factor of n by outputting a single vertex, so that's not great...)

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Sample questions to ask yourself for graph problems:

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One very useful concept that can often be exploited is **low treewidth**. The definition is too complicated for this course, but if your graph can be split into two by removing only a few vertices, and then you can continue the process recursively, then it probably has low treewidth.

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We call problems with algorithms like this **fixed-parameter tractable** (**FPT**). The analogue of NP-hardness in this setting is **W[1]-hardness**.

These can be useful search terms when looking for practical algorithms!

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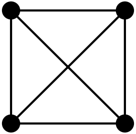

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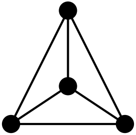

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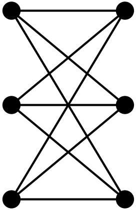

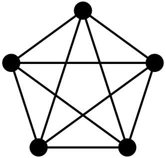

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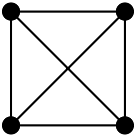

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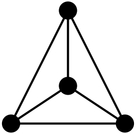

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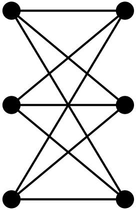

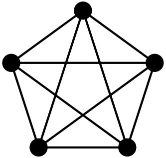

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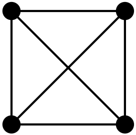

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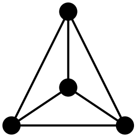

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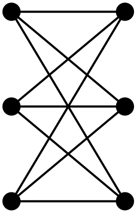

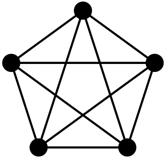

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Of course, if your instances are of size 15–20 you can just use brute force...

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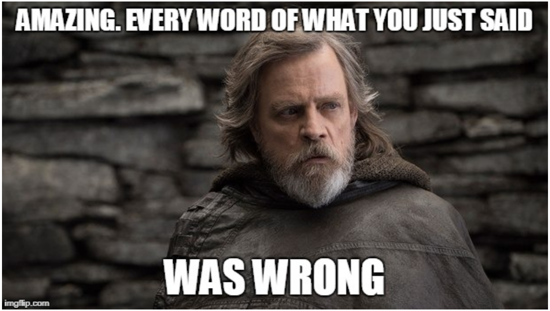

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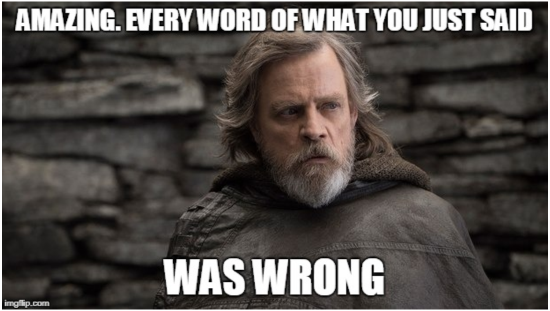

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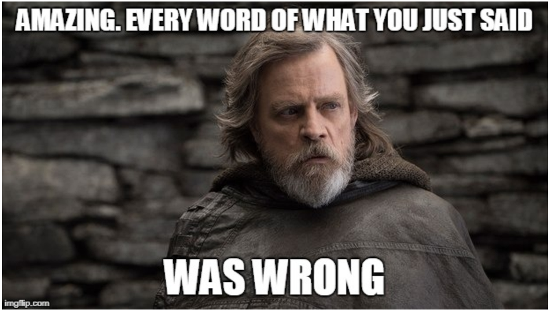

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Clearly "real-world SAT instances" often have some nice properties that solvers are exploiting... but we can't nail down what those properties are. This is a huge open problem right now.

So just give it a try! It might not work, but it's worth checking.